# The deformations of flat affine structures on the two-torus

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**Abstract.** The group action which defines the moduli problem for the deformation space of flat affine structures on the two-torus is the action of the affine group Aff(2) on $\mathbb{R}^2$. Since this action has non-compact stabiliser $\mathrm{GL}(2,\mathbb{R})$, the underlying locally homogeneous geometry is highly non-Riemannian. In this chapter, we describe the deformation space of all flat affine structures on the two-torus. In this context interesting phenomena arise in the topology of the deformation space, which, for example, is *not* a Hausdorff space. This contrasts with the case of constant curvature metrics, or conformal structures on surfaces, which are encountered in classical Teichmüller theory. As our main result on the space of deformations of flat affine structures on the two-torus we prove that the holonomy map from the deformation space to the variety of conjugacy classes of homomorphisms from the fundamental group of the two-torus to the affine group is a local homeomorphism.

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## Contents

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# 1 Introduction

A flat affine structure on a smooth manifold is specified by an atlas with coordinate changes in the group of affine transformations of $\mathbb{R}^n$. A manifold together with such an atlas is called a *flat affine manifold*.  Equivalently, a flat affine manifold is a smooth manifold which has a flat and torsion-free connection on the tangent bundle. A particular class of examples is furnished by Riemannian flat manifolds, but the class of flat affine manifolds is much larger. The study of flat affine manifolds has a long history which can be traced back to the local theory of hypersurfaces and Cartan's projective connections. Global questions were first studied in the context of Bieberbach's theory of crystallographic groups, and they have gained renewed interest in the more general setting by Ehresmann's theory of locally homogeneous spaces, and more recently in Thurston's geometrisation program which shows the importance of locally homogeneous structures in the classification of manifolds.  Flat affine manifolds are *affinely diffeomorphic* if they are diffeomorphic by a diffeomorphism which looks like an affine map in the coordinate charts. The universal covering space of a flat affine manifold admits a local affine diffeomorphism into affine space $\mathbb{A}^n = \mathbb{R}^n$ which is called the *development map*; its image is an open subset of $\mathbb{R}^n$, called the *development image*. The development map and image provide rough invariants for the classification of flat affine manifolds.

Benzécri [15] showed that a closed oriented surface which supports a flat affine structure must be diffeomorphic to a two-torus, thereby confirming in dimension two a conjecture of Chern that the Euler characteristic of a compact flat affine manifold must be zero. The flat affine structures on the two-torus and their development images were partially classified by Kuiper [50] in 1953. The classification was completed by independent work of Furness-Arrowsmith [28] and Nagano-Yagi [61] around 1972. Their works show that the flat affine structures on the two-torus fall into four main classes which have development image the plane $\mathbb{R}^2$, the halfspace, the sector, or the once punctured plane, respectively.

The *moduli space* of flat affine structures is by definition the set of flat affine structures up to affine diffeomorphism. More precisely, the group $\mathrm{Diff}(T^2)$ of all diffeomorphisms of the two-torus $T^2$ acts naturally on the set of flat affine structures on $T^2$. The set of orbits classifies flat affine two-tori up to affine diffeomorphism; it is called the moduli space. The *deformation space* is the

set of all flat affine structures divided by the action of the group $\mathrm{Diff}_0(T^2)$ of diffeomorphisms which are isotopic to the identity. This action classifies flat affine structures on $T^2$ up to isotopy, or equivalently affine two-tori with a marking.

The deformation space has a natural topology, which it inherits from the $C^\infty$-topology on the space of development maps. In this chapter, our aim is to describe the topology of the deformation space $\mathfrak{D}(T^2, \mathbb{A}^2)$ of all flat affine structures on the two-torus. The development process gives, for each flat affine two-torus, a natural homomorphism $h : \mathbb{Z}^2 = \pi_1(T^2) \to \mathrm{Aff}(2)$ of the fundamental group of the torus to the plane affine group $\mathrm{Aff}(2) = \mathrm{Aff}(\mathbb{R}^2)$. This homomorphism is called the *holonomy homomorphism* and it is defined up to conjugacy with an affine map. The holonomy thus gives rise to a continuous open map

$$hol : \mathfrak{D}(T^2, \mathbb{A}^2) \to \mathrm{Hom}(\mathbb{Z}^2, \mathrm{Aff}(2))/\mathrm{Aff}(2)$$

from the deformation space to the space of conjugacy classes of homomorphisms, which is called the *holonomy map*. By a general theorem of Thurston and Weil concerning deformations of *locally homogeneous structures* on manifolds, this map has an *open* image.

As such, the deformation space of flat affine structures is the natural analogue of the Teichmüller space of conformal structures, or, equivalently, constant curvature Riemannian metrics, on surfaces. Its construction is completely analogous to the definition of the Teichmüller space for flat Riemannian metrics on the two-torus, or hyperbolic constant curvature $-1$ metrics on surfaces $M_g$, $g \geq 2$. In these classic situations, both the Teichmüller space $\mathfrak{T}_g$ and its quotient the moduli space are Hausdorff spaces. The Teichmüller space of flat metrics $\mathfrak{T}_1$ is diffeomorphic to $\mathbb{R}^2$, and the Teichmüller space of hyperbolic metrics $\mathfrak{T}_g$, $g \geq 2$, is diffeomorphic to $\mathbb{R}^{6g-6}$. Moreover, the corresponding holonomy map topologically identifies $\mathfrak{T}_g$ with an open subset of the quotient space $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{Isom}(\mathbb{R}^2))/\mathrm{Isom}(\mathbb{R}^2)$, for $g = 1$, or, respectively, a component of the space $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))/\mathrm{PSL}(2, \mathbb{R})$, $g \geq 2$.

Here the analogy with the classical theory breaks down, and neither of these facts are true for the deformation space of flat affine structures. In fact, the group action, which defines the moduli problem for the deformation space of flat affine structures on the two-torus, namely the action of the affine group $\mathrm{Aff}(2)$ on the homogeneous space

$$X = \mathbb{R}^2 = \mathrm{Aff}(2)/\mathrm{GL}(2, \mathbb{R})$$

has non-compact stabiliser $\mathrm{GL}(2, \mathbb{R})$, and therefore the underlying geometry on $X$ is highly non-Riemannian. This is illustrated by the fact that various kinds of flat affine structures, with sometimes strikingly distinct geometric properties, are supported on the two-torus. A fact which can be seen already from the various possible development images for flat affine structures, and

which is also reflected in the structure and topology of the deformation space. Here phenomena arise which are completely different from the case of constant curvature metrics or conformal structures on surfaces.

Another salient difference stems from the fact that the local model of the deformation space of flat affine structures, namely the *character variety* $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{Aff}(2))/\mathrm{Aff}(2)$ arises as a quotient space of an algebraic variety by a *non-reductive* group action. The properties of such actions and their invariant theory are generally poorly understood.

The case of deformation of *complete* flat affine structures bears the closest resemblance to the classical situation. A flat affine structure is called *complete* if the development map is a diffeomorphism, a property which in the Riemannian situation is always guaranteed. A flat affine two-torus is complete if and only if its development image is the affine plane $\mathbb{A}^2$. The deformation space of complete affine structures on the two-torus was studied recently in [2, 7]. It is shown there, for example, that the holonomy map identifies the space of complete structures $\mathfrak{D}_c(T^2, \mathbb{A}^2)$ with a locally closed subspace of the space of homomorphisms $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{Aff}(\mathbb{R}^2))$, and, moreover, the space $\mathfrak{D}_c(T^2, \mathbb{A}^2)$ is homeomorphic to $\mathbb{R}^2$. However, the topology of the moduli space of complete flat affine structures, which, with respect to appropiately chosen coordinates for $\mathfrak{D}_c(T^2, \mathbb{A}^2)$, is homeomorphic to the quotient space of $\mathbb{R}^2$ by the natural action of $\mathrm{GL}(2, \mathbb{Z})$, is highly singular.

This chapter is devoted to the study of the global and local structure of the space of deformations of *all* flat affine structures on the two-torus. The deformation space of all flat affine structures is much larger than the deformation space of complete flat affine structures. Indeed, the deformation space $\mathfrak{D}_c(T^2, \mathbb{A}^2)$ of complete flat affine structures on the two-torus forms a closed two-dimensional subspace in the deformation space of all structures $\mathfrak{D}(T^2, \mathbb{A}^2)$, which itself is a space of dimension four. In the general situation the holonomy map $hol : \mathfrak{D}(T^2, \mathbb{A}^2) \to \mathrm{Hom}(\mathbb{Z}^2, \mathrm{Aff}(2))/\mathrm{Aff}(2)$ for the deformation space of flat affine structures is no longer a homeomorphism onto its image. That is, there exist flat affine structures on the two-torus, which have the same holonomy group and which have dramatically different geometry. (Compare, in particular, Example 5.5 in this chapter.) Moreover, the holonomy image in $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{Aff}(2))$ contains singular orbits for the affine group $\mathrm{Aff}(2)$, which in turn give rise to non-closed points in the deformation space $\mathfrak{D}(T^2, \mathbb{A}^2)$. This also shows that the deformation space is *not* a Hausdorff space. It is a four-dimensional and connected space which has an intricate topology and it supports various substructures arising from the different types of affine flat geometries on the two-torus.

As our main result on the local structure of the space of deformations of flat affine structures we prove in this chapter that *the holonomy map hol is a local homeomorphism onto its image*. That is, at least locally the topology of the deformation space $\mathfrak{D}(T^2, \mathbb{A}^2)$ is fully controlled by the character variety.

We remark that this is *not* a general phenomenon for deformation spaces of locally homogeneous structures, *not even on surfaces*. Indeed, in Appendix B of this chapter, we specify a two-dimensional homogeneous geometry whose deformation space of structures on the two-torus has a holonomy map *hol* which locally near certain structures is a branched covering. Examples of deformation spaces of flat conformal structures on three-dimensional manifolds where the holonomy map *hol* is not locally injective at exceptional points were found previously by Kapovich and are discussed in [44].

The chapter is organized as follows. In Section 2 we give a self-contained proof of Benzécri's theorem which states that a closed orientable flat affine surface is diffeomorphic to the two-torus. In Section 3 we describe the deformation theory of compact locally homogeneous manifolds, including its foundational results and give basic examples. Section 4 discusses several methods to construct flat affine surfaces and introduces the main classes of flat affine structures on the two-torus. In Section 5 we prove the main classification theorem for flat affine structures on the two-torus in detail, including the crucial and nontrivial fact that the development map of such a structure is always a covering map. Finally, in Section 6 we put the pieces together in order to prove that the holonomy map for the deformation space of flat affine structures on the two-torus is a local homeomorphism to the character variety. In addition, Appendix A gives an account on conjugacy classes in $GL(2, \mathbb{R})$ and in its universal covering group. In Appendix B we describe a two-dimensional homogeneous geometry such that the holonomy map for its deformation space of structures on the two-torus is not everywhere a local homeomorphism.

## 2 The theorem of Benzécri

Let $M$ be a closed oriented surface of genus $g$. Then the Gauß-Bonnet theorem [41] expresses the Euler characteristic

$$\chi(M) = 2 - 2g$$

as an integral over the Gauß curvature of any Riemannian metric on $M$. In particular, a flat Riemannian closed surface $M$ has Euler characteristic zero, and therefore it is diffeomorphic to a two-torus. If $M$ is a closed flat affine surface, the Gauß-Bonnet theorem does not apply, since the corresponding flat connection is possibly non-Riemannian. However, the strong topological restriction applies to flat affine surfaces, as well:

**Theorem 2.1** (Benzécri, [15]). *Let $M$ be a closed flat affine surface. Then $M$ has Euler characteristic zero.*

*Proof.* First we remark that the sphere $S^2$ does not admit a flat affine structure. In fact, since $S^2$ is simply connected, the development image of a flat affine structure on $S^2$ would be compact and open in $\mathbb{R}^2$, which is absurd.

Now we assume that $M$ has genus $g$, $g \geq 1$. Let $\mathsf{p} : \tilde{M} \to M$ be the universal covering. Then $\tilde{M}$ is a flat affine manifold which is diffeomorphic to $\mathbb{R}^2$. Moreover, $M$ is obtained by gluing a $4g$-gon $P \subset \tilde{M}$ along its consecutive sides $a_1, b_1, a_1^-, b_1^-, \ldots, a_g, b_g, a_g^-, b_g^-$, with side pairing transformations $g_{a_i}, g_{b_i}$ such that $g_{a_i} a_i^- = a_i$ and $g_{b_i} b_i^- = b_i$. These transformations are subject to the single cycle relation

$$\prod_{i=1,\ldots,g} g_{a_i} g_{b_i} g_{a_i}^{-1} g_{b_i}^{-1} = \mathrm{id}_{\tilde{M}}$$

and generate the discontinuous group of deck transformations of the covering $\mathsf{p} : \tilde{M} \to M$. In particular, the side pairing transformations are affine maps of $\tilde{M}$. Note however that the polygon $P \subset \tilde{M}$ is a closed oriented topological disc with piecewise smooth boundary. (The construction may be carried out, in Euclidean geometry if $g = 0$, respectively hyperbolic geometry, for $g \geq 2$, such that the edges of $P$ are geodesic segments. See, for example, [65].)

Let $\tilde{x}_0$ denote the vertex of $P$ belonging to the sides $a_1$ and $b_g^-$. Let $v \neq 0$ be a tangent vector at $\tilde{x}_0$. Now choose a *non-vanishing* vector field $V$ along the boundary of $P$, such that $V(\tilde{x}_0) = v$, and, furthermore, such that $V$ restricted to $a_i$ (resp. $b_i$) is related to $V$ restricted to $a_i^-$ (resp. $b_i^-$) by the corresponding side pairing transformation. (We can obtain such a $V$ by constructing vector fields along the closed curves $\mathsf{p}\, a_i$, $\mathsf{p}\, b_i$ which coincide at $x_0 = \mathsf{p}(\tilde{x}_0)$.) Next we extend $V$ to a vector field $X$ on $P$, which has an isolated singularity in the interior of $P$.

The following and other similar pictures illustrate convergence of development maps in the deformation space of flat affine structures on the two-torus. Each development map gives rise to a tiling of an open domain in affine space which is deformed with the change of development maps.
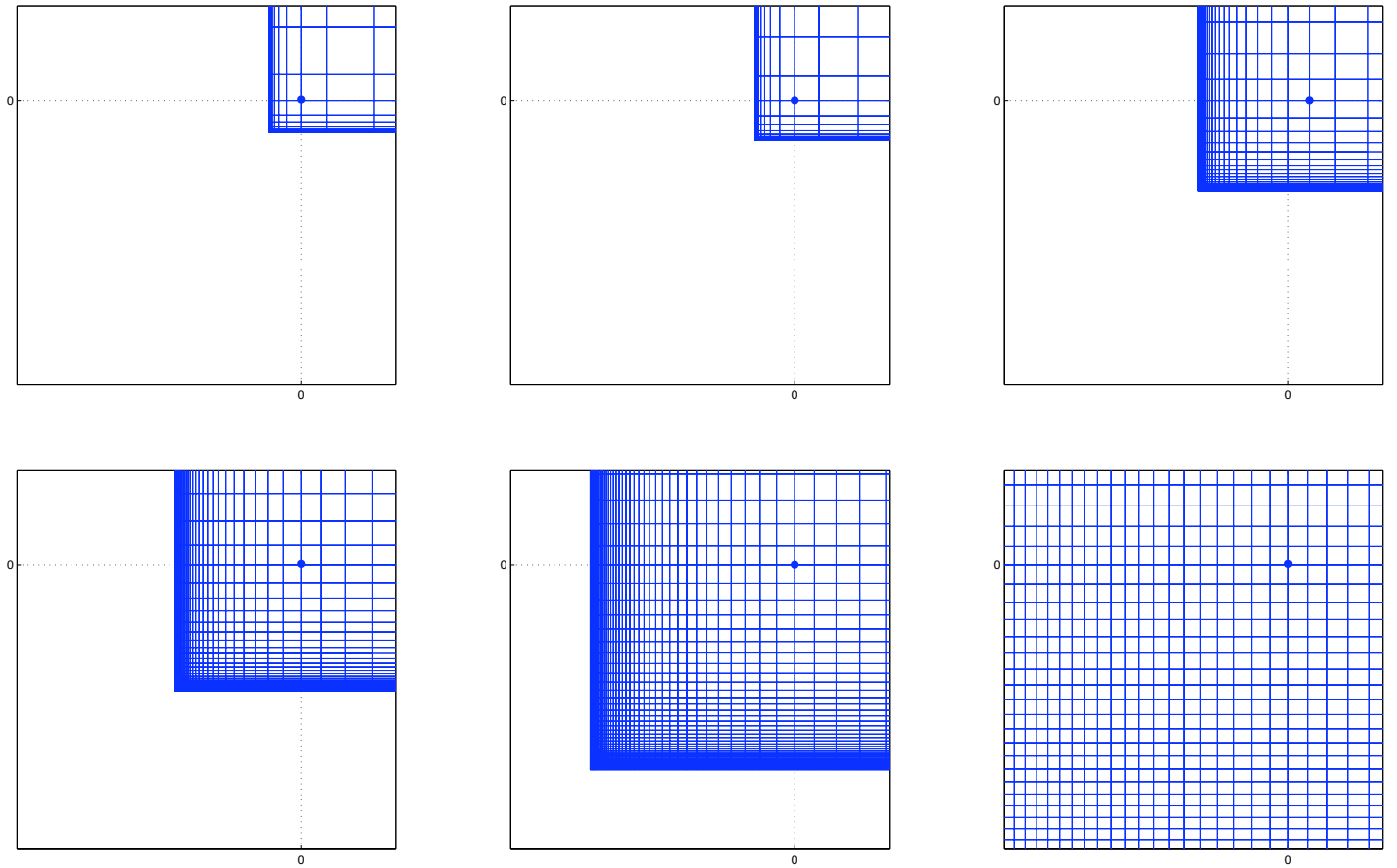


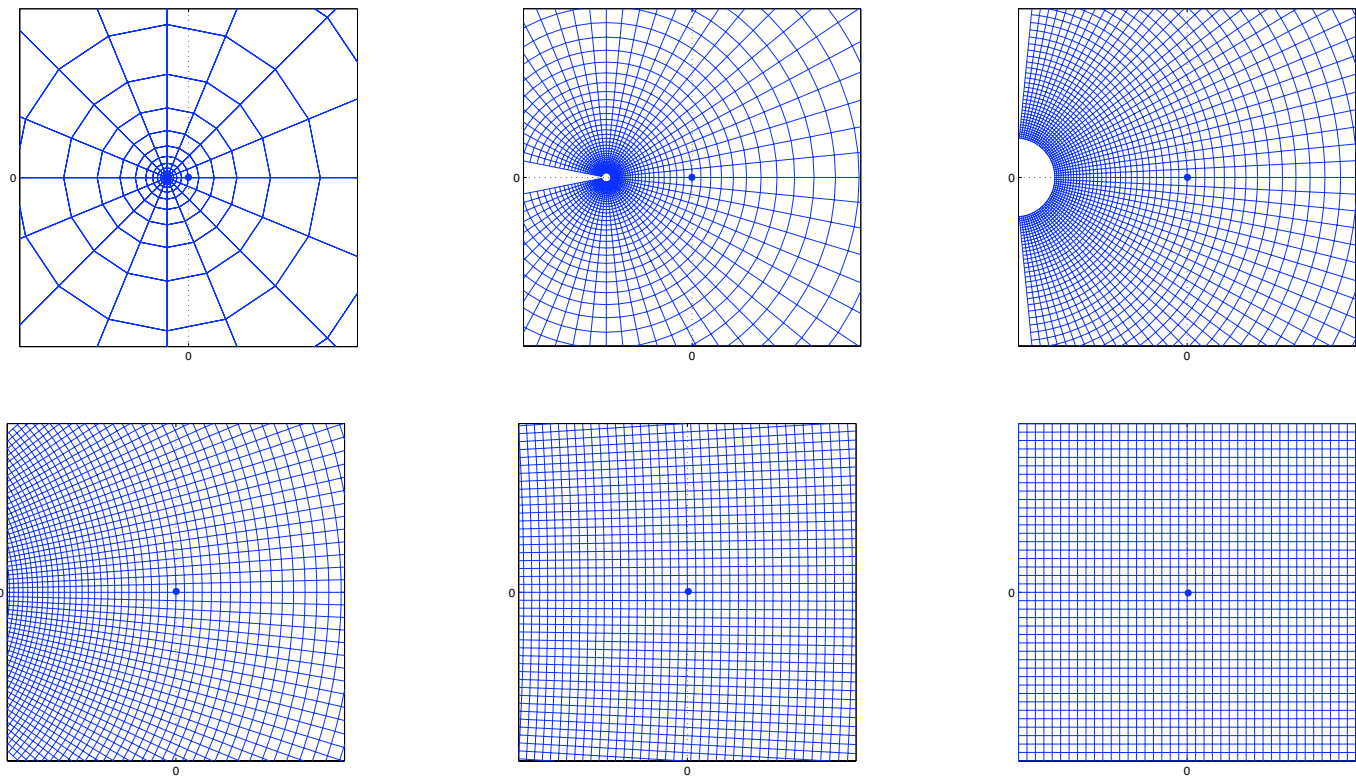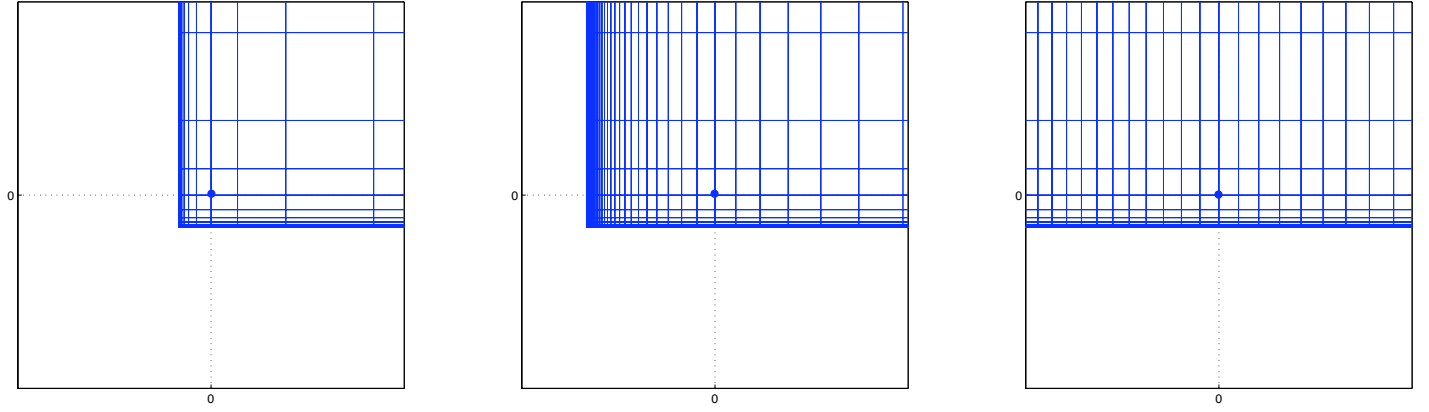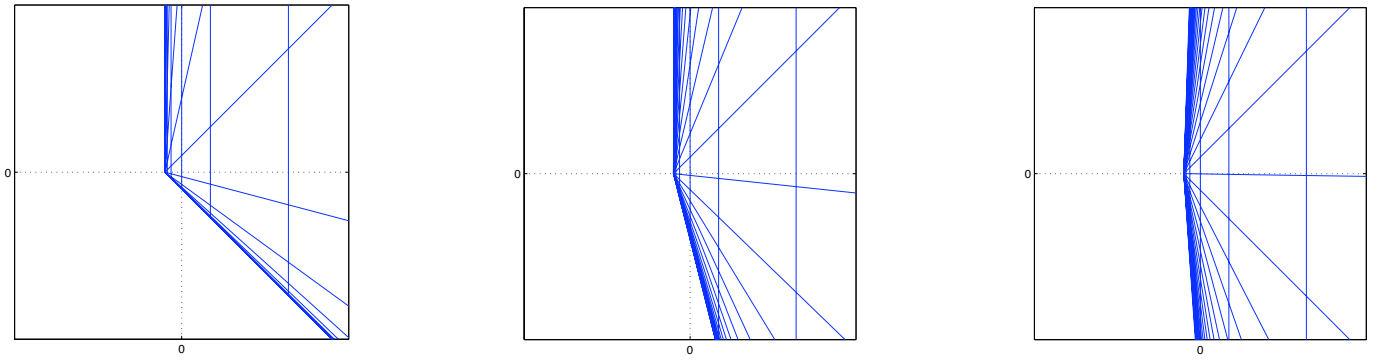Figure 1. Tiled sectors approaching the standard plane.

Figure 2. Tiled punctured planes approaching the standard plane.

Figure 3. Tiled sectors approaching a halfplane of type $C_2$.



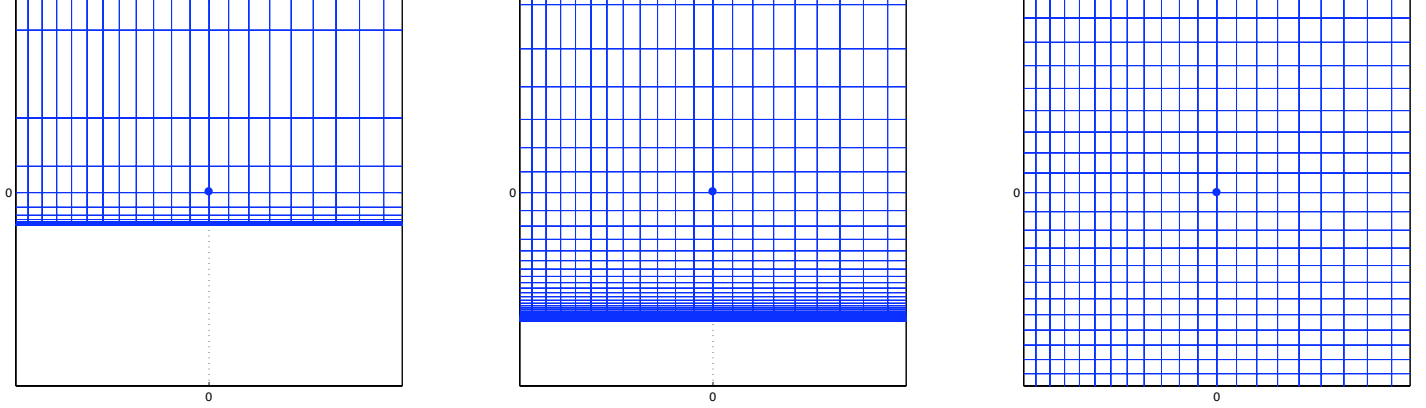Figure 4. Tiled sectors approaching a halfplane of type $C_1$.

Figure 5. Tiled halfplanes approaching the standard plane.

The *index* of the vector field $X$ at the singularity may be calculated as the *turning number* of the restriction of $X$ to the positively traversed boundary of $P$, see [41]. For this, recall that the turning number $\tau(V) \in \mathbb{Z}$ of a closed non-vanishing vector field $V : I \to \mathbb{A}^2 - 0$ is defined by the equation

$$\tau(V)\,2\pi = \theta(1) - \theta(0),$$

where $\theta : I \to \mathbb{R}$ is any lift of of the map $I \to S^1$, $t \mapsto V(t)/|V(t)|$. Now, since the flat affine manifold $\tilde{M}$ is simply connected, $\tilde{M}$ has a global parallelism which identifies each tangent space $T_x\tilde{M}$ with $T_{\tilde{x}_0}\tilde{M}$. Therefore, we can choose any scalar product in $T_{\tilde{x}_0}\tilde{M}$ to compute the index of $X$ by the above formula.

Since the side pairing maps $g_{a_i}$ and $g_{b_i}$ are affine transformations of $\tilde{M}$, they preserve antipodality of any two vectors $V(s)$ and $V(t)$. This implies that the turn of $V$ restricted to the positively traversed curve $a_1 b_1 a_1^- b_1^-$ is less than $2\pi$. And consequently, $|\tau(V)| < g$.

Our construction implies that the vector field $X$ on $P$ projects to a vector field on $M$. Therefore, by the Poincaré-Hopf theorem [41, 57], the index of $X$ equals the Euler characteristic $\chi(M)$ of $M$. We thus obtain the estimate

$$|\chi(M)| = |2 - 2g| \ < g \ .$$

This implies $g = 1$.                                                         $\square$

Benzécri's theorem was generalised by Milnor [56] to the more general

**Theorem 2.2.** *Let $E$ be a flat rank two vector-bundle over a closed orientable surface $M_g$, $g \geq 1$, then $|\chi(E)| < g$.*

Here, $\chi(E)$ denotes the evaluation of the Euler-class $e(E) \in H^2(M, \mathbb{Z})$ on the fundamental homology class of $M$. In case of the tangent bundle $E = TM_g$, the equality

$$\chi(TM_g) = \chi(M_g)$$

(see [58, Section 11]) implies Benzécri 's theorem. Wood [76] interpreted Milnor's result in the context of circle-bundles. See [37] for a recent survey on Benzécri 's theorem, Milnor's inequality and related topics.

**Generalizations to higher dimensions**   Weak analogues of the Milnor-Wood estimate for the Euler-class of higher-dimensional manifolds were subsequently given by Sullivan [69] and Smillie in his doctoral thesis [67]. See also [18] for a recent contribution in this realm.

The Chern conjecture asserts that any compact flat affine manifold should have Euler characteristic zero. Kostant and Sullivan [49] observed that every compact *and* complete flat affine manifold has Euler characteristic zero. There are some additional affirmative results under various assumptions on the holonomy group, see for example [31]. The original conjecture, however, remains a difficult open problem.

Another fruitful generalization of the Milnor-Wood inequality concerns the representation theory of surface groups into higher-dimensional simple Lie groups, see [19] for a survey.

# 3 Locally homogeneous structures and their deformation spaces

Let $M$ be a smooth manifold and fix a universal covering space $\mathsf{p} : \tilde{M} \to M$. Let $X$ be a homogeneous space for the Lie group $G$, on which $G$ acts effectively. The manifold $M$ is said to be locally modeled on $(X, G)$ if $M$ admits an atlas of charts with range in $X$ such that the coordinate changes are locally restrictions of elements of $G$. A maximal atlas with this property is called an $(X, G)$-*structure* on $M$. The manifold $M$ together with an $(X, G)$-structure is called an $(X, G)$-*manifold*, or *locally homogeneous space* modeled on $(X, G)$. A map between two $(X, G)$-manifolds is called an $(X, G)$-*map* if it coincides with the action of an element of $G$ in the local charts. If the $(X, G)$-map is a diffeomorphism it is called an $(X, G)$-*equivalence* and accordingly the two manifolds are called $(X, G)$-*equivalent*.

## 3.1 $(X, G)$-manifolds, development map and holonomy

Every $(X, G)$-manifold comes equipped with some extra structure, called the development and the holonomy. Via the covering projection $\mathsf{p} : \tilde{M} \to M$ the universal covering space of the $(X, G)$-manifold $M$ inherits a unique $(X, G)$-structure from $M$. We fix $x_0 \in M$, and a local $(X, G)$-chart at $x_0$. The corresponding *development map* of the $(X, G)$-structure is the $(X, G)$-map

$$D : \tilde{M} \to X$$

which is obtained by analytic continuation of the local chart.

For every $(X, G)$-equivalence $\Phi$ of $\tilde{M}$, there exists a unique element $h(\Phi) \in G$ such that

$$D \circ \Phi = h(\Phi) \circ D \ . \tag{3.1}$$

The fundamental group $\pi_1(M) = \pi_1(M, x_0)$ acts on $\tilde{M}$ via deck transformations. This induces the *holonomy homomorphism*

$$h : \pi_1(M, x_0) \longrightarrow G$$

which satisfies

$$D \circ \gamma = h(\gamma) \circ D \ , \quad \text{for all } \gamma \in \pi_1(M, x_0). \tag{3.2}$$

Note that, after the choice of the development map (which corresponds to a choice of a germ of an $(X, G)$-chart in $x_0$ and also the choice of a lift $\tilde{x}_0 \in \tilde{M}$ of $x_0$), the holonomy homomorphism $h$ is well defined. Therefore, the $(X, G)$-structure on $M$ determines the *development pair* $(D, h)$ up to the action of $G$, where $G$ acts by left-composition on $D$, and by conjugation on $h$. Specifying a development pair is equivalent to constructing an $(X, G)$-structure on $M$:

**Proposition 3.1.** *Every local diffeomorphism $D : \tilde{M} \to X$ which satisfies (3.2), for some $h : \pi_1(M, x_0) \to G$, defines a unique $(X, G)$-structure on $M$, and every $(X, G)$-structure on $M$ arises in this way.*

**3.1.1 Compactness and completeness of $(X, G)$-manifolds** An important special case arises if the development map is a diffeomorphism. Recall the following definition:

**Definition 3.2 (Proper actions).** A discrete group $\Gamma$ is said to act *properly discontinuously* on $X$ if, for all compact subsets $\kappa \subseteq X$, the set

$$\Gamma_\kappa = \{ \gamma \in \Gamma \mid \gamma \kappa \cap \kappa \neq \emptyset \}$$

is finite. More generally, if $\Gamma$ is a locally compact group, and $\Gamma_\kappa$ is required to be compact, then the action is called *proper*.

**Example 3.3** (($X, G$)**-space forms**). Let $\Gamma$ be a group of $(X, G)$-equivalences acting properly discontinuously and freely on $X$. Then $X/\Gamma$ is a manifold which inherits an $(X, G)$-structure from $X$. If $X$ is simply connected the identity map of $X$ is a development map for $X/\Gamma$.

In general, if the development map is a covering map onto $X$, the $(X, G)$-manifold $M$ will be called *complete*.

Simple examples (cf. the Hopf tori in Example 4.8) show that compactness of $M$ does not imply completeness.

**Example 3.4** (**Compactness and completeness**). If $G$ acts properly on $X$ then every *compact* $(X, G)$-manifold is complete.

In general, the relation between the properties of the $G$-action on $X$, and the completeness properties of compact $(X, G)$-manifolds is only vaguely understood. See [20] for a striking contribution in this direction in the context of flat affine manifolds. Further discussion of $(X, G)$-geometries and the properties of the development process may be found in [24, 71].

It may well happen that an $(X, G)$-geometry does not admit (non-finite) proper actions (see [46, 51, 43]) or no compact $(X, G)$-manifolds at all [13].

**Example 3.5** (**Calabi-Marcus phenomenon**). Let $\mathbb{A}^2 - 0$ be the once-punctured affine plane. It is easily observed that every discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$ which acts properly on $\mathbb{A}^2 - 0$ must be finite, see Figure 6. This is called the *Calabi-Markus phenomenon*.



Figure 6. Dynamics of a hyperbolic rotation and a shearing acting on $\mathbb{A}^2 - 0$

It follows that the homogeneous space $(\mathbb{A}^2 - 0, \mathrm{SL}(2, \mathbb{R}))$ has only quotients by finite groups. Therefore, a complete space modeled on $(\mathbb{A}^2 - 0, \mathrm{SL}(2, \mathbb{R}))$ cannot be compact. In fact, we will remark in Example 3.11 below that there do not exist compact $(\mathbb{A}^2 - 0, \mathrm{SL}(2, \mathbb{R}))$-manifolds at all.

**Prominent** ($X, G$)**-structures on surfaces**    Let $M_g$ denote an orientable surface of genus $g$. In the context of this paper, the following $(X, G)$-structures play a prominent role.

**Example 3.6.**

  (1) $(\mathbb{S}^2, \O(2))$, spherical geometry, $g = 0$.

  (2) $(\mathbb{R}^2, \mathrm{E}(2))$, plane Euclidean geometry, $g = 1$.

  (3) $(\mathbb{H}^2, \mathrm{PSL}(2, \mathbb{R}))$, plane hyperbolic geometry, $g \geq 2$.

  (4) $(\mathbb{R}^2, \mathrm{Aff}(2))$, plane affine geometry, $g = 1$.

  (5) $(\mathbb{P}^2(\mathbb{R}), \mathrm{PSL}(3, \mathbb{R}))$, plane projective geometry, $g \geq 0$.

Every compact orientable surface of genus $g \geq 2$ supports hyperbolic structures. Also every compact surface supports a projective structure, see [21]. By Benzécri's theorem the only compact surfaces which support a flat affine structure are the two-torus and the Klein bottle. The classification of flat affine structures on the two-torus was completed in the 1970's, see Section 5 of this article. Subsequently, Bill Goldman in his undergraduate thesis [30] classified projective structures on the two-torus in 1977.

Note that every Euclidean or hyperbolic compact surface is complete (compare Example 3.4). The majority of flat affine structures on the two-torus are *not* complete but the development map of a flat affine structure on the two-torus is always a covering onto its image (see Theorem 5.1). The development map of a projective structure on a surface may not even be a covering [21].

**3.1.2 $(X, G)$-subgeometries** We may relate different locally homogeneous geometries by inclusion as follows.

**Definition 3.7.** Let $(X, G)$ and $(X', G')$ be homogeneous spaces and $\rho : G' \to G$ a homomorphism together with a $\rho$-equivariant local diffeomorphism $o : X' \to X$. Then we say that $(X', G')$ *is subjacent to* or *a subgeometry of* $(X, G)$. The subgeometry is called *full* if the map $o$ is surjective onto $X$. The subgeometry is called a *covering of geometries* if $o : X' \to X$ is a regular covering map with group of deck transformations precisely the kernel of $\rho$.

If $(X', G')$ is a subgeometry of $(X, G)$ then $X'$ is an $(X, G)$-manifold with development map $o : X' \to X$. Note also that $o$ is a covering map onto its image, since $o$ is an equivariant map of homogeneous spaces. The group $G'$ then acts as a group of $(X, G)$-equivalences of $X'$, so that $X'$ is, in fact, a homogeneous $(X, G)$-manifold.

**Example 3.8.** Let $\mathsf{p} : \widetilde{\mathbb{A}^2 - 0} \to \mathbb{A}^2 - 0$ be the universal covering of the once-punctured affine plane $\mathbb{A}^2 - 0$, and $\widetilde{\mathrm{GL}}(2, \mathbb{R}) \to \mathrm{GL}(2, \mathbb{R})$ the universal covering group of $\mathrm{GL}(2, \mathbb{R})$. Then the subgeometry

$$\mathsf{p} : (\widetilde{\mathbb{A}^2 - 0}, \widetilde{\mathrm{GL}}(2, \mathbb{R})) \longrightarrow (\mathbb{A}^2 - 0, \mathrm{GL}(2, \mathbb{R}))$$

is a full subgeometry and, indeed, it is a covering of geometries.

If $(X', G')$ is a subgeometry of $(X, G)$ then, in particular, *every $(X', G')$-manifold with development map $D'$ inherits naturally an $(X, G)$-manifold structure with development map $o \circ D'$.*

This observation provides a useful tool to construct $(X, G)$-manifolds. Assume, for instance, that $G'$ acts properly on $X'$ and $\Gamma' \leq G'$ is a discrete subgroup. Then $\Gamma' \backslash X'$ is an $(X', G')$-manifold which inherits an $(X, G)$-structure via $o$. The following special case is of particular importance:

**Definition 3.9** (**Étale $(X, G)$-representations**). If $G'$ acts on $X'$ with finite stabilizer then an inclusion of geometries $\rho : G' \to G$ as above is called an *étale representation* of $G'$ into $(X, G)$.

If $\rho$ is étale with open orbit $\rho(G')x_0$ the group manifold $G'$ inherits via the orbit map

$$o : G' \to X , \; g' \mapsto \rho(g')x_0$$

a natural $(X, G)$-structure which is invariant by left-multiplication of $G'$. In particular, if $\Gamma' \leq G'$ is a discrete subgroup then the coset space $\Gamma' \backslash G'$ inherits an $(X, G)$-manifold structure.

**Example 3.10** (**Geometries subjacent to the punctured plane**). The affine automorphism group of the once punctured affine plane $\mathbb{A}^2 - 0$ is the linear group $\mathrm{GL}(2, \mathbb{R})$. The homogeneous geometry $\left(\mathbb{A}^2 - 0, \mathrm{GL}(2, \mathbb{R})\right)$ has full subgeometries $\left(\mathbb{A}^2 - 0, \mathrm{GL}(1, \mathbb{C})\right)$ and $(\mathbb{A}^2 - 0, \mathrm{SL}(2, \mathbb{R}))$. Note that the first one arises from an étale affine representation of the abelian Lie group $\mathbb{R}^2$. Further subgeometries, which are not full, are defined by the abelian étale Lie subgroups $\mathsf{C}_1$ and $\mathsf{B}$, which are listed in (2) and (3) of Example 4.2. Of course, all these homogeneous spaces define particular subgeometries of plane affine geometry, as well.

**3.1.3 Existence of compact forms** A compact manifold $M$, which is locally modeled on $(X, G)$, will be called a *compact form* for $(X, G)$. Given a homogeneous space $(X, G)$, it is possibly a difficult problem to decide if it has compact form.

**Example 3.11** ($(\mathbb{A}^2 - 0, \mathrm{SL}(2, \mathbb{R}))$ **has no compact form**). By the Calabi-Markus phenomenon (see Example 3.5), $(\mathbb{A}^2 - 0, \mathrm{SL}(2, \mathbb{R}))$ has only quotients by finite groups. Since $(\mathbb{A}^2 - 0, \mathrm{SL}(2, \mathbb{R}))$ is a subgeometry of plane affine geometry, Benzécri's theorem (Theorem 2.1) and the classification of flat affine structures with development image $\mathbb{A}^2 - 0$ (see Theorem 5.1 ) imply the stronger result that *there is no compact locally homogeneous surface modeled on the homogeneous space $(\mathbb{A}^2 - 0, \mathrm{SL}(2, \mathbb{R}))$.*

On the contrary, the spaces $(\mathbb{A}^2 - 0, \mathrm{GL}(2, \mathbb{R}))$ and $(\mathbb{A}^2 - 0, \mathrm{GL}(1, \mathbb{C}))$ evidently have complete compact forms. For example, every lattice $\Gamma \leq \mathrm{GL}(1, \mathbb{C})$ acts properly discontinuously and freely on $\mathbb{A}^2 - 0$, which thus gives rise to a compact flat affine manifold $\mathbb{A}^2 - 0 \,/\, \Gamma$.

Benzécri's theorem implies that every orientable compact form of the space $(\mathbb{A}^2 - 0, \mathrm{GL}(2, \mathbb{R}))$ is diffeomorphic to the two-torus, and in particular it has abelian fundamental group $\mathbb{Z}^2$.

**Example 3.12** (**Compact forms of** $(\mathbb{A}^2 - 0, \mathrm{GL}(2, \mathbb{R}))$). The classification theorem asserts that the development map of a flat affine structure on the two-torus is a covering map onto the development image (cf. Proposition 5.8). In particular, every compact locally homogeneous surface modeled on the homogeneous spaces $(\mathbb{A}^2 - 0, \mathrm{GL}(1, \mathbb{C}))$ or $(\mathbb{A}^2 - 0, \mathrm{GL}(2, \mathbb{R}))$ is *diffeomorphic to the two-torus* and either it is complete (which is always true for $(\mathbb{A}^2 - 0, \mathrm{GL}(1, \mathbb{C}))$-structures) or its development image is a sector of halfspace in $\mathbb{A}^2$ (see Section 4.1).

In Section 4.3, we describe the construction of all compact $(\mathbb{A}^2 - 0, \mathrm{GL}(2, \mathbb{R}))$-manifolds which are *complete*. The classification theorem for all structures, including the non-complete case, is stated in Section 5.2.

## 3.2  Convergence of development maps

The space of $(X, G)$-development maps for the manifold $M$ is the set

$$\mathrm{Dev}(M) = \mathrm{Dev}(M, X, G)$$

of all local $C^\infty$-diffeomorphisms

$$D : \tilde{M} \to X$$

which, for some $h \in \mathrm{Hom}(\pi_1(M), G)$, and, for all $\gamma \in \pi_1(M)$, satisfy

$$D \circ \gamma = h(\gamma) \circ D.$$

We endow the space of development maps with the compact $C^\infty$-topology. In this topology, a sequence of smooth maps converges if and only if *it and all its derivatives* (computed in local coordinate charts) converge uniformly on the compact subsets of $\tilde{M}$. In particular, $\mathrm{Dev}(M)$ thus becomes a Hausdorff second countable topological space.

**3.2.1 Convergence of holonomy** Let $M$ be compact. Then $\pi_1(M)$ is finitely generated, and we equip $\mathrm{Hom}(\pi_1(M), G)$ with the topology of pointwise convergence. Then the map

$$\mathsf{hol} : \mathrm{Dev}(M) \to \mathrm{Hom}(\pi_1(M), G) \,, \ D \mapsto h$$

is continuous, since $G$ has the $C^\infty$-topology of maps on $X$. The main theorem on deformations of $(X, G)$-structures (see Theorem 3.15 below) asserts that a small deformation of holonomy induces a deformation of development maps. That is, the map hol admits local sections. By compactness of $M$, the convergence of development maps is controlled on a fundamental domain and by the holonomy.

**Fact 3.13** (Holonomy determines convergence). Let $U \subset \tilde{M}$ be an open subset with compact closure such that $\mathsf{p}(U) = M$, where $\mathsf{p} : \tilde{M} \to M$ is the universal covering. Then a sequence of development maps $D_i \in \mathrm{Dev}(M)$ with holonomy $h_i$ converges to a development map $D$ if and only if the restrictions of $D_i$ to $U$ converge to $D$ and the homomorphisms $h_i$ converge to the holonomy $h$ of $D$.

A particular property of the $C^\infty$-topology is that it does not control the behavior of maps outside compact sets. This allows for possibly unexpected phenomena:

**Example 3.14** (**Openness of embeddings fails**). Let $\mathrm{Dev}_e(M)$ be the subset of development maps which are injective. Let $\mathcal{K} \subset \tilde{M}$ be a compact fundamental domain for the action of $\pi_1(M)$. By [39, Chapter 2, Lemma 1.3], the set of development maps which are injective on $\mathcal{K}$ is open with respect to the $C^1$-topology. In particular, it is open with respect to the $C^\infty$-topology. However, the global behavior of development maps is controlled by the holonomy. Therefore, even if $M$ is compact $\mathrm{Dev}_e(M)$ may not be an open subset in $\mathrm{Dev}(M)$. On the two-torus there are injective development maps in $\mathrm{Dev}(T^2, \mathbb{A}^2, \mathrm{Aff}(2))$ which contain a non-trivial covering map in every small neighborhood. This is even true for the development of the standard translation structure, cf. Section 6.4.1 and Figure 7.

**3.2.2 Deformation of development maps**  If $M$ is compact then, as observed by Thurston [72], building on earlier work of Weil [74], a small deformation of holonomy in the space of homomorphisms $\mathrm{Hom}(\pi_1(M), G)$ induces a deformation of $(X, G)$-development maps. Before stating the theorem precisely, we discuss the

**Action of diffeomorphisms of $M$ on development pairs.**  Let $x_0 \in M$ and $\tilde{x}_0 \in \tilde{M}$, $\mathsf{p}(\tilde{x}_0) = x_0$, be basepoints and $\Phi \in \mathrm{Diff}(M, x_0)$ a basepoint preserving diffeomorphism with lift $\tilde{\Phi} \in \mathrm{Diff}(\tilde{M}, \tilde{x}_0)$. The group $\mathrm{Diff}(M, x_0)$ of basepoint preserving diffeomorphisms then acts on development pairs, by mapping $D \in \mathrm{Dev}(M)$ to $D \circ \tilde{\Phi}$. We let $\mathrm{Diff}_1(M, x_0)$ denote the subgroup of $\mathrm{Diff}(M, x_0)$ consisting of diffeomorphisms which are homotopic to the identity by a basepoint preserving homotopy, and $\mathrm{Diff}_0(M, x_0)$ the identity component of $\mathrm{Diff}(M, x_0)$ (that is, the subgroup of elements which are isotopic to
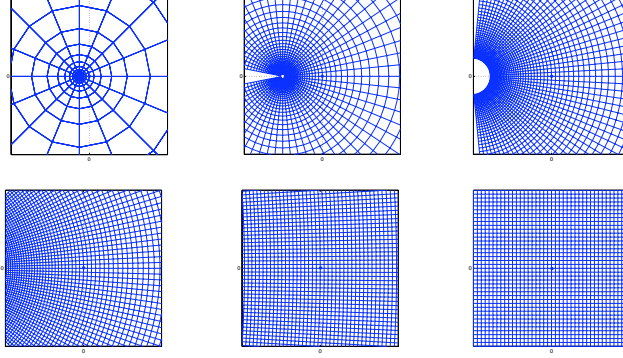
Figure 7. The quotient of a Hopf torus deforms to a translation torus.

the identity). Then the action of $\mathrm{Diff}_1(M, x_0)$ and its subgroup $\mathrm{Diff}_0(M, x_0)$ on the set of development maps $\mathrm{Dev}(M)$ leaves the holonomy invariant, since $\mathrm{Diff}_1(M, x_0)$ acts trivially on $\pi_1(M, x_0)$.

See [24, 31, 53, 16] for more detailed discussion of the following:

**Theorem 3.15** (Deformation theorem, Thurston et al.). *Let $M$ be a compact manifold. Then the induced map*

$$\mathsf{hol}: \ \mathrm{Diff}_0(M, x_0) \backslash \mathrm{Dev}(M) \longrightarrow \mathrm{Hom}(\pi_1(M), G) \qquad (3.3)$$

*which associates to a development map its holonomy homomorphism is a local homeomorphism.*

The theorem states that the map $\mathsf{hol} : \mathrm{Dev}(M) \longrightarrow \mathrm{Hom}(\pi_1(M), G)$ is continuous and open. In addition, $\mathsf{hol}$ locally admits *continuous* sections. Such a section is called a *development section*. More specifically, it is proved (see below) that *every convergent sequence of holonomy maps lifts to a convergent sequence of development maps*, and two nearby development maps with identical holonomy are isotopic by a basepoint preserving diffeomorphism. Therefore, a sequence of points in the quotient space $\mathrm{Diff}_0(M, x_0) \backslash \mathrm{Dev}(M)$ is convergent if and only if there exists a corresponding lifted sequence of development maps which converges.

The main idea in the proof of Theorem 3.15 due to Weil [74] is easy to grasp. Here we sketch the construction of the development section in the particular case of flat affine two-tori. In addition, we consider only tori which are obtained by gluing polygons in the plane (cf. Section 4.2). A similar approach is also valid for non-homogeneous tori which are obtained as quotients of the universal

covering affine manifold of $\mathbb{A}^2 - 0$ (cf. Section 4.3), and, in fact, in the general case of arbitrary $(X, G)$-manifolds, compare [53, 74]. A somewhat different approach to this result is explained in [31] and the recent survey [36] on locally homogeneous manifolds.

*Proof of Theorem 3.15.* Let $M$ be a flat affine two-torus and $D_o : \mathbb{R}^2 \to \mathbb{A}^2$, $h_o : \mathbb{Z}^2 \to \mathrm{Aff}(2)$ a development pair for $M$. Let $h_\epsilon \in \mathrm{Hom}(\mathbb{Z}^2, \mathrm{Aff}(2))$, $\epsilon \geq 0$, be a small deformation of $h_o$. To obtain the development section, we construct a curve of development maps $D_\epsilon$ with holonomy $h_\epsilon$, which converges to $D_o$ in the compact $C^\infty$-topology.

For this, we assume that the development pair of $M$ is represented as the identification space of a polygon $\mathcal{P}$ in affine space. In fact, as explained in [2, Section 2], $\mathcal{P}$ can be chosen to be a quadrilateral contained in $\mathbb{A}^2$, which is glued along its sides by the generators $\gamma_1$, $\gamma_2$ of $\pi_1(T^2) = \mathbb{Z}^2$ using the holonomy images $h_o(\gamma_i) \in \mathrm{Aff}(2)$. The generators satisfy cycle relations and certain gluing conditions. Next we fix a diffeomorphism of the standard unit square in $\mathbb{R}^2$ with $\mathcal{P}$. Using $h$, this extends $\pi_1(T^2)$-equivariantly to a smooth covering

$$\mathbb{R}^2 \; \to \; \bar{X} = (P \times \Gamma)\big/ \sim \; ,$$

where the identification space $\bar{X}$ is a flat affine manifold which is obtained as the disjoint union of the polygons $\gamma P$, $\gamma \in \Gamma$, glued along their edges as determined by the side pairings $h_o(\gamma_i)$. Here $\Gamma = h_o(\mathbb{Z}^2)$ is the holonomy group of $M$. Moreover, the inclusion $\mathcal{P} \to \mathbb{A}^2$ extends to a development map

$$\bar{D} : \bar{X} \to \mathbb{A}^2 \; .$$

The composition of both maps yields the desired development map $D : \mathbb{R}^2 \to \mathbb{A}^2$ with holonomy $h_o$. The space $\bar{X}$ is the holonomy covering space of $M$, see [2, Proposition 2.1] for a detailed account.

The development section $D_\epsilon$ may now be obtained in a similar manner. In fact, for small $\epsilon > 0$, $\mathcal{P}$ can be deformed continuously to a quadrilateral $\mathcal{P}_\epsilon$, which satisfies the gluing conditions with respect to $h_\epsilon$. (See Figure 8 for an illustration.) This gives rise to a series of identification spaces $\bar{X}_\epsilon = (P \times \Gamma_\epsilon)\big/ \sim_\epsilon$, and corresponding development maps $D_\epsilon : \mathbb{R}^2 \to \mathbb{A}^2$ with holonomy $h_\epsilon$. By the above Fact 3.13, the developments maps $D_\epsilon$ converge to $D_o$ in the $C^\infty$-topology. $\qquad \square$

The above construction of the development section is illustrated in Figures 8 and 9.

### 3.2.3 Topological rigidity of development maps
Although local rigidity holds by the deformation theorem, it may fail globally. If the map hol in (3.3) is *not* injective (as happens in the case of flat affine two-tori, see the basic Examples 4.9, 4.15 and also Section 6 for further discussion), there do exist
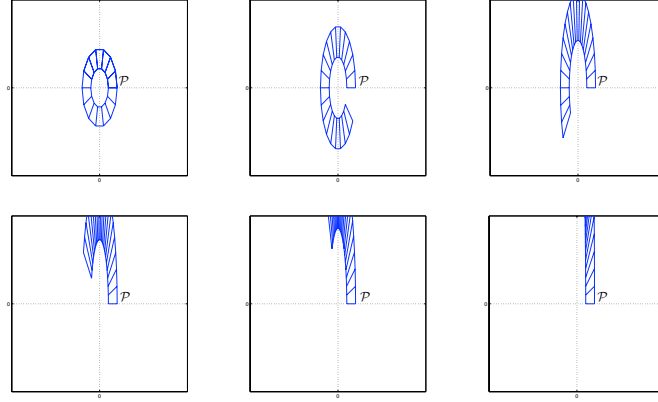
Figure 8. The fundamental polygon $\mathcal{P}$ and its development deform with the holonomy.
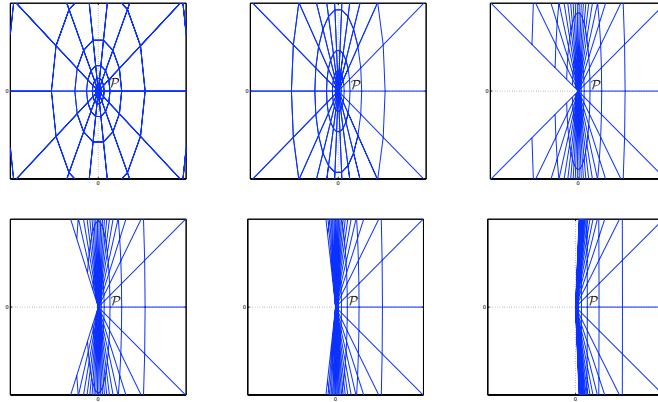


Figure 9. A family of development maps for the once-punctured plane $\mathbb{A}^2 - 0$ collapses to the development process of an affine half-plane.

non-isomorphic $(X, G)$-manifolds with the same holonomy homomorphism $h$. On the contrary, if the domain of discontinuity $\Omega$ for the holonomy group $h(\Gamma)$, $\Gamma = \pi_1(M)$, on $X$ is large then the development is uniquely determined. This is the case, for example, if $\Omega = X$ and $h(\Gamma)$ is the holonomy of a compact complete $(X, G)$-manifold.

**Example 3.16** (**Discontinuous holonomy**). Let $D : \tilde{M} \to X$ be the development map for an $(X, G)$-structure on the compact manifold $M$ with holonomy homomorphism $h$. If $h(\Gamma)$ acts properly discontinuously and freely with compact quotient on $X$ then $D$ is a covering map onto $X$. (In fact, $D$ is a covering, since the local diffeomorphism on compact manifolds $M \to X/h(\Gamma)$ induced by $D$ is a covering map.) It follows that every other development map $D' : \tilde{M} \to X$ with holonomy homomorphism $h$ is of the form $D' = D \circ \Phi$, where $\Phi \in \text{Diff}(\tilde{M})$ is a diffeomorphism which centralizes the deck transformation group $\Gamma$.

A more involved argument allows to show that $D$ is determined by $h(\Gamma)$ if the Hausdorff dimension of $X - \Omega$ is small, see [38].

**Example 3.17.** The development maps of compact $(\widetilde{\mathbb{A}^2 - 0}, \widetilde{\text{GL}}^+(2, \mathbb{R}))$-forms are rigid, see Section 5.2, Theorem 5.6. We remark that the domain of discontinuity for the holonomy group of such a manifold can be a proper open subset of $\widetilde{\mathbb{A}^2 - 0}$.

## 3.3 Deformation spaces of $(X, G)$-structures

Let $\mathfrak{S}(M) = \mathfrak{S}(M, X, G)$ denote the set of all $(X, G)$-structures on $M$. The group $\text{Diff}(M)$ of all diffeomorphisms of $M$ acts naturally on this set such that two $(X, G)$-structures are in the same orbit if and only if they are $(X, G)$-equivalent. The set of all $(X, G)$-structures on $M$ up to $(X, G)$-equivalence is called the *moduli space* $\mathfrak{M}(M) = \mathfrak{M}(M, X, G)$ of $(X, G)$-structures.

**Definition 3.18.** The *deformation space* for $(X, G)$-structures on $M$ is the quotient space

$$\mathfrak{D}(M) = \mathfrak{D}(M, X, G) = \mathfrak{S}(M, X, G)/\text{Diff}_1(M)$$

of equivalence classes of $(X, G)$-structures up to *homotopy*.

Thus, two $(X, G)$-structures define the same point in $\mathfrak{D}(M)$ if they are equivalent by an $(X, G)$-equivalence which is homotopic to the identity of $M$. The moduli space $\mathfrak{M}(M)$ is the quotient space of the deformation space $\mathfrak{D}(M)$ by the group of homotopy classes of diffeomorphisms of $M$.

**Remark 3.19.** There is some inconsistency in the literature about the definition of the deformation space. Many authors define $\mathfrak{D}(M)$ to be the space of structures up to *isotopy*. If $M$ is a surface (two-dimensional manifold) two homotopic diffeomorphisms are isotopic, by classical results of Dehn, Nielsen, and Baer (see for example [68]). Therefore, in this case, these two definitions coincide. The corresponding fact fails in higher dimensions, even for tori, see [42].

We observe that the Lie group $G$ acts by left-composition on the space of development maps. This action is continuous *and free*, and the set of $(X, G)$-structures naturally identifies with the quotient by the action of $G$, that is,

$$\mathfrak{S}(M, X, G) = G \backslash \mathrm{Dev}(M, X, G) .$$

Indeed, if $g \in G$ and $D \in \mathrm{Dev}(M, X, G)$ is a development map then $g \circ D$ is another development map for the same $(X, G)$-structure on $M$. This exhibits the deformation space as a double quotient space

$$\mathfrak{D}(M, X, G) = G \backslash \mathrm{Dev}(M, X, G) / \mathrm{Diff}_1(M) .$$

The $C^\infty$-topology on the set of $(X, G)$-structures is the quotient topology inherited from $\mathrm{Dev}(M, X, G)$. (Thurston [71, Chapter 5] also gives a direct description of the topology on $\mathfrak{S}(M)$ in terms of convergence of sets of local charts which define the elements of $\mathfrak{S}(M)$, see [24, 1.5.1].) The deformation space and the moduli space carry the quotient topology inherited from the set of $(X, G)$-structures.

### 3.3.1 Orientation components of the deformation space

Let $X$ be a $G$-space which is orientable. We let $G^+$ denote the normal subgroup of orientation preserving elements of $G$. Now assume that $M$ is an $(X, G)$-manifold which is orientable. We fix an orientation for $M$. Then there is a disjoint decomposition

$$\mathrm{Dev}(M, X, G) = \mathrm{Dev}^+(M, X, G) \cup \mathrm{Dev}^-(M, X, G), \qquad (3.4)$$

where $\mathrm{Dev}^+(M, X, G)$ and $\mathrm{Dev}^-(M, X, G)$ denote the closed (and open) subspaces which consist of orientation preserving and of orientation reversing development maps, respectively.

Since $M$ is orientable, the components of the decomposition (3.4) are preserved by the action of $\mathrm{Diff}_1(M, x_0)$ on development maps. Furthermore, the action of $G^+$ on development maps preserves the components. Therefore, the deformation space $\mathfrak{D}(M, X, G^+)$ decomposes into two disjoint open and closed subsets, the *orientation components*,

$$\mathfrak{D}(M, X, G^+) = \mathfrak{D}^+(M, X, G) \cup \mathfrak{D}^-(M, X, G) . \qquad (3.5)$$

Note that every orientation reversing element of $G$ exchanges the orientation components of $\mathrm{Dev}(M, X, G)$ and therefore also of $\mathfrak{D}(M, X, G^+)$. Hence, if

$G$ contains orientation reversing elements then the subgeometry $(X, G^+) \to (X, G)$ induces a homeomorphism

$$\mathfrak{D}^+(M, X, G) \approx \mathfrak{D}(M, X, G) \,.$$

**3.3.2 The topology of the deformation space** The following classical and fundamental example gives a role model for the investigation of the properties of deformation spaces for locally homogeneous structures.

**Example 3.20** (Teichmüller space $\mathfrak{T}_g$). Let $G^+ = \mathrm{PSL}(2, \mathbb{R})$ be the group of orientation preserving isometries of the hyperbolic plane $\mathbb{H}_2$ and $M = M_g$ a surface of genus $g$, $g \geq 2$. By the uniformization theorem, the Teichmüller space $\mathfrak{T}_g$ of conformal structures on a surface $M_g$, $g \geq 2$, may be considered as the deformation space of constant curvature $-1$ metrics, that is,

$$\mathfrak{T}_g = \mathfrak{D}^+(M_g, \mathbb{H}_2, \mathrm{PSL}(2, \mathbb{R})) \,.$$

The space $\mathfrak{T}_g$ is homeomorphic to $\mathbb{R}^{6g-6}$. Recall that the *mapping class group*

$$\mathrm{Map}_g = \mathrm{Diff}^+(M_g)/\mathrm{Diff}_0(M_g) \cong \mathrm{Out}^+(\Gamma_g)$$

is the group of isotopy classes of orientation preserving diffeomorphisms of a surface. This group acts properly discontinuously on $\mathfrak{T}_g$, and the moduli space of conformal structures

$$\mathfrak{M}(M_g) = \mathfrak{T}_g / \mathrm{Map}_g$$

is a Hausdorff space. (See, for example, [1, 23, 65] and other chapters of this handbook [19, 35]).

In general, however, the topology on the moduli space and the deformation space can be highly singular, as we can see, in particular, from Examples 3.26 and 3.29 below. The local properties of the deformation space are reflected in the character variety $\mathrm{Hom}(\pi_1(M), G)/G$, which is the space of conjugacy classes of representations of $\pi_1(M)$ into $G$.

**The holonomy map on the deformation space** Since $\mathfrak{S}(M, X, G)$ has the quotient topology from development maps, the holonomy (3.3) induces a continuous map

$$\overline{\mathsf{hol}} : \mathfrak{S}(M, X, G) \longrightarrow \mathrm{Hom}(\pi_1(M), G)/G \,,$$

which gives rise to the map

$$hol : \mathfrak{D}(M) \longrightarrow \mathrm{Hom}(\pi_1(M), G)/G \,. \tag{3.6}$$

The continuous map *hol* associates to a homotopy class of $(X, G)$-structures on $M$ the corresponding conjugacy class of its holonomy homomorphism $h$.

By the deformation theorem (Theorem 3.15), *hol* is furthermore an open map.

The map *hol* thus encodes a good picture of the topology on $\mathfrak{D}(M)$:

**Example 3.21** (Teichmüller space $\mathfrak{T}_g$ is a cell). The holonomy image of hyperbolic structures on $M_g$, $g \geq 2$, is the subspace $\mathrm{Hom}_c(\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))$ of the space $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))$ which consists of injective homomorphisms with discrete image. The space $\mathrm{Hom}_c(\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))$ has two connected components [33]. The components $\mathrm{Hom}_c^+(\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))$ and $\mathrm{Hom}_c^-(\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))$ arise from the orientation of development maps. The group $\mathrm{PSL}(2, \mathbb{R})$ acts freely and properly (by conjugation) on

$$\mathrm{Hom}_c^+(\Gamma_g, \mathrm{PSL}(2, \mathbb{R})) \,,$$

the quotient space being homeomorphic to $\mathbb{R}^{6g-6}$. (See, for example, [65, Theorem 9.7.4]). By completeness of hyperbolic structures on $M_g$, every development map is a diffeomorphism. The topological rigidity of development maps (cf. Example 3.16) implies that the induced map

$$\mathfrak{T}_g = \mathfrak{D}^+(M_g, \mathbb{H}_2) \xrightarrow{hol} \mathrm{Hom}_c^+(\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))/\mathrm{PSL}(2, \mathbb{R})$$

is a homeomorphism.

In general, it seems difficult to decide if the map *hol* is a local homeomorphism, as well. Indeed, Kapovich [44] constructed examples of deformation spaces such that the map *hol* is *not* everywhere a local homeomorphism. We construct such a counterexample for the deformation space of a two-dimensional geometric structure on the two-torus in Appendix B. In the case of flat affine two-tori though, we shall show that *hol* is a local homeomorphism (see Section 6.5).

**The induced map of a subgeometry**   Let $o : (X', G') \to (X, G)$ be a subgeometry with $\rho : G' \to G$ the associated homomorphism (see Section 3.1.2). There is an associated map

$$\mathrm{Dev}(M, X') \longrightarrow \mathrm{Dev}(M, X) \,, \ D' \mapsto D = o \circ D' \tag{3.7}$$

and a map on homomorphisms

$$\mathrm{Hom}(\pi_1(M), G') \longrightarrow \mathrm{Hom}(\pi_1(M), G) \,, \ h' \mapsto h = \rho \circ h' \,,$$

where $h = \mathsf{hol}(D)$. These maps allow to relate the deformation spaces in a commutative diagram of the form

$$
\begin{array}{ccc}
\mathfrak{D}(M, X') & \xrightarrow{\;\;hol\;\;} & \mathrm{Hom}(\pi_1(M), G')/G' \\
\downarrow & & \downarrow \\
\mathfrak{D}(M, X) & \xrightarrow{\;\;hol\;\;} & \mathrm{Hom}(\pi_1(M), G)/G \;.
\end{array}
\qquad (3.8)
$$

Note that the properties of the induced map $\mathfrak{D}(M, X') \to \mathfrak{D}(M, X)$ can vary wildy with various types of subgeometries. In general, the induced map need not be injective nor surjective.

Recall the notion of covering of geometries from Definition 3.7. We shall require the following lemma:

**Lemma 3.22.** *If $o : (X', G') \to (X, G)$ is a covering of geometries then the induced map on deformation spaces*

$$
\mathfrak{D}(M, X') \longrightarrow \mathfrak{D}(M, X)
$$

*is a homeomorphism.*

*Proof.* Indeed, since $o$ is a covering the above map (3.7), $D' \mapsto D$, on development maps descends to a $\mathrm{Diff}(M)$-equivariant map on the sets of structures

$$
\mathfrak{S}(M, X') \longrightarrow \mathfrak{S}(M, X)
$$

which is a homeomorphism. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

**3.3.3 The topology on the space of $(X, G)$-structures** The topology on the space $\mathfrak{S}(M, X, G)$ is rather well behaved. In fact, $\mathfrak{S}(M, X, G)$ is a Hausdorff and metrizable topological space. This can be seen by representing an $(X, G)$-structure on $M$ as an integrable higher order structure in the sense of Ehresmann (cf. [45, Section I.8]). We discuss two important examples now:

**Example 3.23** ($\mathfrak{S}(M, \mathbb{H}_2, \mathrm{PSL}(2, \mathbb{R}))$). The space of hyperbolic structures on a surface $M$ is homeomorphic to the space of hyperbolic (constant curvature $-1$) Riemannian metrics with the $C^\infty$-topology on the space of Riemannian metrics. It can also be equipped with the structure of a contractible Fréchet manifold, see [23]. Similarly, the space $\mathfrak{S}(M, \mathbb{R}^2, \mathrm{E}(2))$ of flat Euclidean structures is homeomorphic to the space of flat Riemannian metrics on $M$.

In the case of flat affine structures, the action of the affine group on development maps admits a global slice:

**Example 3.24** ($\mathfrak{S}(M, \mathbb{A}^n)$ is Hausdorff). Let $\mathrm{Dev}(M, \mathbb{A}^n)$ be the set of development maps for flat affine structures on $M$. We choose a base frame $E_{x_0}$ on $\mathbb{A}^n$ and a frame $F_{\tilde{m}_0}$ on $\tilde{M}$, respectively, and let

$$\mathrm{Dev}_f(M, \mathbb{A}^n) = \mathrm{Dev}_f(M, F_{\tilde{m}_0}, E_{x_0})$$

denote the set of frame preserving development maps. Since $\mathrm{Aff}(n)$ acts simply transitively on the frame bundle of $\mathbb{A}^n$, there is a well defined continuous retraction $\mathrm{Dev}(M, \mathbb{A}^n) \to \mathrm{Dev}_f(M, \mathbb{A}^n)$, and, in fact, there is a homeomorphism

$$\mathrm{Dev}(M, \mathbb{A}^n) \approx \mathrm{Aff}(n) \times \mathrm{Dev}_f(M, \mathbb{A}^n) \ .$$

This shows that the quotient $\mathrm{Dev}(M, \mathbb{A}^n)/\mathrm{Aff}(n)$ is homeomorphic to the subspace $\mathrm{Dev}_f(M, \mathbb{A}^n)$ and the affine group $\mathrm{Aff}(n)$ acts properly on the set of development maps. In particular, the space of flat affine structures $\mathfrak{S}(M, \mathbb{A}^n)$ is a Hausdorff space.

Another way to understand the topology on $\mathfrak{S}(M, \mathbb{A}^n)$ is to identify flat affine structures with *flat torsion free connections* on the tangent bundle of $M$. These form a space of sections of a quotient of the bundle of 2-frames over $M$, see [45, Proposition IV.7.1]. In Section 6.1 of this chapter we employ this approach to study flat affine structures on the two-torus.

**3.3.4 The subspace of complete $(X, G)$-structures** Let $\mathfrak{D}_c(M)$ denote the subset of the deformation space $\mathfrak{D}(M)$ which consists of complete $(X, G)$-space forms (that is, the subspace corresponding to development maps which are diffeomorphisms). We denote with $\mathrm{Hom}_c(\pi_1(M), G)$ the set of all injective homomorphisms $\pi_1(M) \to G$, such that the image acts properly discontinuously on $X$. We call $\mathrm{Hom}_c(\pi_1(M), G)$ the *set of discontinuous homomorphisms*. The holonomy homomorphisms belonging to the elements of $\mathfrak{D}_c(M)$ form an open subset of $\mathrm{Hom}_c(\pi_1(M), G)$. In fact, by the rigidity of development maps belonging to discontinuous holonomy homomorphisms (cf. Example 3.16), a small deformation of holonomy, which remains in the domain of discontinuous homomorphisms, lifts to a deformation of complete $(X, G)$-manifold structures on $M$. Therefore, Theorem 3.15 implies that the restricted map

$$\mathsf{hol}: \ \mathrm{Diff}_0(M, x_0)\backslash \mathrm{Dev}_c(M) \longrightarrow \mathrm{Hom}_c(\pi_1(M), G)$$

is a local homeomorphism. Then the following result is easily observed (see also [3]):

**Theorem 3.25.** *Let $M$ be a smooth compact manifold such that the natural homomorphism $\mathrm{Diff}(M)/\mathrm{Diff}_1(M) \to \mathrm{Out}(\pi_1(M))$ is injective. Then the induced map*

$$hol: \mathfrak{D}_c(M) \longrightarrow \mathrm{Hom}_c(\pi_1(M), G)/G$$

*is a homeomorphism onto its image.*

Note that the assumptions of the theorem are satisfied, for example, if $X$ is contractible.

**Example 3.26** (Complete flat affine structures on $T^2$). The holonomy image of development maps for complete flat affine structures on the two-torus is $\mathrm{Hom}_c(\mathbb{Z}^2, \mathrm{Aff}(2))$, that is, it consists of all injective homomorphisms with properly discontinuous image. As is shown in [2, Section 4.4], this is a locally closed subset of $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{Aff}(2))$, defined by algebraic equalities and inequalities, and it has two connected components. Moreover, the conjugation action of the group $\mathrm{Aff}(2)$ on $\mathrm{Hom}_c(\mathbb{Z}^2, \mathrm{Aff}(2))$ is orbit equivalent to its restriction to the subgroup $\mathrm{GL}(2, \mathbb{R})$. The latter group acts freely and properly on $\mathrm{Hom}_c(\mathbb{Z}^2, \mathrm{Aff}(2))$ and the quotient space is homeomorphic to $\mathbb{R}^2$. Since

$$\mathfrak{D}_c(T^2, \mathbb{A}^2) \xrightarrow{\;hol\;} \mathrm{Hom}_c(\mathbb{Z}^2, \mathrm{Aff}(2))/\mathrm{Aff}(2)$$

is a homeomorphism, the deformation space of complete flat affine structures $\mathfrak{D}_c(T^2, \mathbb{A}^2)$ is homeomorphic to $\mathbb{R}^2$. As is shown in [2, 7], natural coordinates can be chosen such that the action of $\mathrm{Map}^+(T^2) = \mathrm{SL}(2, \mathbb{Z})$ on $\mathfrak{D}_c(T^2, \mathbb{A}^2)$ corresponds to the standard representation of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{R}^2$.

**3.3.5 Deformation of lattices (A. Weil, 1962)** Let $G$ be a simply connected Lie group and $\Gamma_o \leq G$ a cocompact lattice. We put

$$M_o = G/\Gamma_o \, ,$$

where $\Gamma_o$ acts by left-multiplication on the universal cover $\tilde{M}_o = G$ of $M_o$. Let $(X, G_L) = (G, G_L)$ be the homogeneous geometry which is defined by the action of $G$ on itself by left-multiplication. Since the action of $G$ on itself is proper, every $(G, G_L)$-manifold is complete. Hence

$$\mathfrak{D}(M_o, G) = \mathfrak{D}_c(M_o, G)$$

and the holonomy image of $\mathfrak{D}(M_o, G)$ is contained in the space of lattice homomorphisms

$$\mathrm{Hom}_L(\Gamma_o, G) = \{\rho : \Gamma_o \hookrightarrow G \mid \rho(\Gamma_o) \text{ is a lattice in } G\} \, .$$

We call the space of conjugacy classes of lattice homomorphisms

$$\mathfrak{D}_L(\Gamma_o, G) = \mathrm{Hom}_L(\Gamma_o, G)/G$$

the deformation space of the lattice $\Gamma_o$. The holonomy map

$$hol : \mathfrak{D}(M_o, G) \longrightarrow \mathfrak{D}_L(\Gamma_o, G) \tag{3.9}$$

therefore locally embeds $\mathfrak{D}(M_o, G)$ as an open (and closed) subspace of the deformation space of $\Gamma_o$. This is the original setup which is studied in the

seminal paper [74] by André Weil. Fundamental results on the nature of the involved spaces $\mathrm{Hom}_L(\Gamma_o, G)$ and $\mathfrak{D}_L(\Gamma_o, G)$ are obtained in the foundational papers [74, 75, 73], see also [16]. For a recent contribution in the context of solvable Lie groups $G$, see [8]; the examples which are constructed in [8, Section 2.3] show that there exist deformation spaces of the form $\mathfrak{D}(M_o, G)$, which have infinitely many connected components.

**Rigidity of lattices and action of the automorphism group of** $G$   Note that the group $\mathrm{Aut}(G)$ of automorphisms of $G$ has natural actions on the space of development maps $\mathrm{Dev}(M_o, G)$ and on $\mathrm{Hom}_L(\Gamma_o, G)$. Indeed, let $\phi \in \mathrm{Aut}(G)$, and $D : G \to G$ a development map for an $(G, G_L)$-structure on $M_o$ with holonomy $\rho \in \mathrm{Hom}_L(\Gamma_o, G)$. Then the composition

$$\phi \circ D : G \to G$$

is a development map with holonomy $\phi \circ \rho$. These actions descend to actions on $\mathfrak{D}(M_o, G)$, $\mathfrak{D}_L(\Gamma_o, G)$ respectively, such that (3.9) becomes an equivariant map.

**Example 3.27** (Rigid lattices). A lattice $\Gamma_o$ is called *rigid* in $G$ if $\mathrm{Aut}(G)$ acts transitively on $\mathfrak{D}_L(\Gamma_o, G)$. For example, lattices in nilpotent Lie groups $G$, or lattices in simple Lie groups $G$ not locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$ are rigid, see [59]. In these two cases we then have identities

$$\mathfrak{D}(M_o, G) \xrightarrow{\approx} \mathfrak{D}_L(\Gamma_o, G) \approx \mathrm{Aut}(G)/\mathrm{Inn}(G) ,$$

for any lattice $\Gamma_o \leq G$. Here, $\mathrm{Inn}(G)$ denotes the group of inner automorphisms of $G$.

More generally, we call a lattice $\Gamma_o$ *smoothly rigid*, if the holonomy map (3.9) is a homeomorphism, that is, if $\mathfrak{D}(M_o, G) = \mathfrak{D}_L(\Gamma_o, G)$. For example, lattices in solvable Lie groups are smoothly rigid, by a theorem of Mostow; but there do exist solvable Lie groups which admit non-rigid lattices. See [59], or [8] and the references therein for specific examples.

The deformation spaces of the form $\mathfrak{D}(M_o, G)$ play an important role in the analysis of general deformation spaces, since many geometric structures arise from étale representations. An illustrative example is given by the stratification of the space of deformations of flat affine structure on the two-torus which is studied in detail in Section 6.4.

**The induced map of an étale representation**   Let $G'$ be a simply connected Lie group and $\Gamma_o \leq G'$ a cocompact lattice. We put $M_o = G'/\Gamma_o$. Let us assume for simplicity that $\Gamma_o$ is smoothly rigid as well. Now let $(X, G)$ be a homogeneous space and $\rho : G' \to G$ be an étale representation (see Definition

3.9). Then the orbit map which is associated to an open orbit of $G'$ defines a subgeometry

$$o : (G', G'_L) \to (X, G)$$

which in turn gives rise to a map (3.8) of deformation spaces

$$\mathfrak{D}(M_o, G') \longrightarrow \mathfrak{D}(M_o, X, G) \;,$$

that is, we obtain a map

$$\mathfrak{D}_L(\Gamma_o, G') = \mathrm{Hom}_L(\Gamma_o, G')/G' \to \mathfrak{D}(M_o, X, G) \;.$$

This map factors over the action of the normalizer $\mathrm{N}_G(\rho)$ of $\rho(G')$ in $G$, that is, we have an induced map

$$\mathrm{Hom}_L(\Gamma_o, G')/\mathrm{N}_G(\rho) \to \mathfrak{D}(M_o, X, G) \;.$$

We remark that, if $\Gamma_o$ is rigid in $G'$ then

$$\mathrm{Hom}_L(\Gamma_o, G')/\mathrm{N}_G(\rho) = \mathrm{Aut}(G')/N \;,$$

where $N \leq \mathrm{Aut}(G')$ denotes the image of $\mathrm{N}_G(\rho)$ in $\mathrm{Aut}(G')$.

**3.3.6 Dynamics of the $G$-action on $\mathrm{Hom}(\Gamma, G)$** In Examples 3.21 and 3.26 above, the map *hol* is a homeomorphism, and the corresponding deformation spaces are Hausdorff. These properties hold in particular if the holonomy image in $\mathrm{Hom}(\Gamma, G)/G$ is obtained as a quotient by a *proper* group action. In fact, if $G$ acts properly (and freely) on the image of *hol*, then, by the slice theorem (cf. [62]), the projection map $\mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\Gamma, G)/G$ admits a section near every holonomy homomorphism. It then follows from Theorem 3.15 that *hol* $: \mathfrak{D}(M) \to \mathrm{Hom}(\Gamma, G)/G$ is a local homeomorphism.

**Example 3.28** (Subvariety of stable points). If $G$ is a reductive linear algebraic group, then, by a general fact on representations of such groups, there exists a Zariski-open subset of stable points in $\mathrm{Hom}(\Gamma, G)$, where $G$ acts properly. Recall that, for any representation of $G$ on a vector space, or any action of $G$ on an affine variety $V$, a point $x \in V$ is called *stable* if the orbit $Gx$ is closed and $\dim Gx = \dim G$. The set of stable points may be empty though. For the action of $G$ on $\mathrm{Hom}(\Gamma, G)$ it is non-empty if there are points $\rho \in \mathrm{Hom}(\Gamma, G)$ such that $\rho(\Gamma)$ is sufficiently dense in $G$. In the specific context where $\Gamma$ is abelian (or solvable), $\mathrm{Hom}(\Gamma, G)$ has no stable points (as follows from [60, Theorem 1.1]). See [60, 31] for further discussion of these facts and for some applications.

One cannot expect $\mathfrak{D}(M)$ to be a Hausdorff space, in general. In fact, the image of *hol* in $\mathrm{Hom}(\pi_1(M), G)/G$ may contain non-closed points. In this

situation also $\mathfrak{D}(M)$ has non-closed points. The following example is due to Bill Goldman:

**Example 3.29** (Non-closed points in $\mathfrak{D}(T^2, \mathbb{A}^2)$). Let

$$A_\epsilon = \left( \begin{array}{cc} \lambda & \epsilon \\ 0 & \lambda \end{array} \right) \ , \ \text{where } \lambda > 1.$$

Then $M_\epsilon = \langle A_\epsilon \rangle \backslash \mathbb{A}^2 - 0$ is a flat affine two-torus, which has an infinite cyclic holonomy group generated by $A_\epsilon$ (see also Example 4.12). Let $\rho_\epsilon$ denote a corresponding holonomy homomorphism for $M_\epsilon$. Since the $A_\epsilon$, $\epsilon \neq 0$, are all conjugate elements of $\mathrm{GL}(2, \mathbb{R})$, the closure of the $\mathrm{GL}(2, \mathbb{R})$-orbit of $\rho_1 \in \mathrm{Hom}(\mathbb{Z}^2, \mathrm{GL}(2, \mathbb{R}))$ contains the holonomy homomorphism $\rho_0$. Therefore, the orbit $[\rho_1]$ is not closed in $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{Aff}(2))/\mathrm{Aff}(2)$. By Corollary 6.10, $M_1$ defines a non-closed point in the deformation space.

Observe that $\rho_0$ is the holonomy of the Hopf torus $\mathcal{H}_\lambda$. By Theorem 3.15 there exists a corresponding family of development maps $D_\epsilon$ with holonomy $\rho_\epsilon$ which converges to the development map of the Hopf torus $M_0 = \mathcal{H}_\lambda$. We observe that these development maps belong to affine structures which are isotopically equivalent to the tori $M_\epsilon$. Hence, the closure of $M_1$ in the deformation space contains the Hopf torus $\mathcal{H}_\lambda$.

(To see explicitly how the development maps for the tori $M_\epsilon$ converge to the Hopf torus in the deformation space, we may use the constructions in Section 4.3 in this chapter. In fact, we construct $M_\epsilon$ as a quotient space $M_\epsilon = \mathcal{T}_{A_\epsilon, \mathrm{id}, 2}$ of $\widetilde{\mathbb{A}^2 - 0}$, as in Example 4.15. Then we deform the development $D = D_0$ of $M_0$ as in the proof of Theorem 3.15 to obtain a sequence of development maps $D_\epsilon : \widetilde{\mathbb{A}^2 - 0} \to \mathbb{A}^2$ for $\mathcal{T}_{A_\epsilon, \mathrm{id}, 2}$ which converges to $D_0$.)

**3.3.7 Dynamics of the $\mathrm{Diff}_0(M)$-action on $(X, G)$-structures** In favorable cases, the topology on $\mathfrak{D}(M)$ may be determined by constructing slices for the action of $\mathrm{Diff}_0(M)$ on $\mathfrak{S}(M, X, G)$. The study of the action of $\mathrm{Diff}_0(M)$ on $\mathfrak{S}(M, X, G)$ may then be used to deduce information on the topology (diffeomorphism groups carry the $C^\infty$-topology) of $\mathrm{Diff}_0(M)$, or, vice versa, on the topology of $\mathfrak{S}(M, X, G)$. The theory of slices for action of diffeomorphism groups on spaces of Riemannian metrics was developed by Palais and Ebin [22]. Recall that a continuous action of $\mathrm{Diff}(M)$ on a space $\mathcal{S}$ is called proper if the map $\mathrm{Diff}(M) \times \mathcal{S} \to \mathcal{S} \times \mathcal{S}$, $(g, s) \mapsto (g \cdot s, s)$ is proper. If the action is proper, the quotient space is Hausdorff (cf. [17, III, Section 4.2]).

**Example 3.30** ($\mathrm{Diff}(M_g)$ acts properly). The group of diffeomorphisms of a closed surface $\mathrm{Diff}(M_g)$ acts properly on the space of conformal structures $\mathfrak{S}(M_g, \mathbb{H}_2, \mathrm{PSL}(2, \mathbb{R}))$, if $g \geq 1$. In particular, the identity component

$\mathrm{Diff}_0(M_g)$ acts properly and freely. Moreover, the projection map

$$\mathfrak{S}(M_g, \mathbb{H}_2, \mathrm{PSL}(2,\mathbb{R})) \to \mathfrak{T}_g$$

is a trivial $\mathrm{Diff}_0(M_g)$-principal bundle. Since $\mathfrak{S}(M_g, \mathbb{H}_2, \mathrm{PSL}(2,\mathbb{R}))$ and $\mathfrak{T}_g$ are contractible, this implies at once that *the group* $\mathrm{Diff}_0(M)$ *is contractible.* These results were shown in [23, Section 5 D].

Similar results hold also for the space $\mathfrak{T}_1$ of conformal structures (flat Riemannian metrics) on the two-torus. In fact, $\mathrm{Diff}(T^2)$ acts properly on the space $\mathfrak{S}(T^2, \mathbb{R}^2, E(2))$, and the moduli space of such structures is a Hausdorff space. However, the action of $\mathrm{Diff}_0(T^2)$ is not free, since every flat Riemannian structure on $T^2$ has $S^1 \times S^1$ acting as a group of isometries. In this situation, we may replace $\mathrm{Diff}(T^2)$ with the subgroup $\mathrm{Diff}(T^2, x_0)$. Indeed, $S^1 \times S^1$ is a deformation retract of $\mathrm{Diff}_0(T^2)$ and the group $\mathrm{Diff}_0(T^2, x_0)$ is contractible (cf. [23]).

In general, the action of $\mathrm{Diff}_1(M)$ on a space of structures $\mathfrak{S}(M, X, G)$ need not be free neither proper, as we show in the following examples.

**Action of** $\mathrm{Diff}(T^2)$ **on the space of flat affine structures** In the case of flat affine structures on the two-torus, the action of $\mathrm{Diff}_0(T^2)$ on the set of all affine structures $\mathfrak{S}(T^2, \mathbb{A}^2)$ is not proper, for otherwise $\mathfrak{D}(T^2, \mathbb{A}^2)$ would be a Hausdorff space. But, in fact, as we show in Example 3.29, $\mathfrak{D}(T^2, \mathbb{A}^2)$ has singularities.

An interesting in-between case arises when restricting to the subspace $\mathfrak{S}_c(T^2, \mathbb{A}^2)$ of *complete* flat affine structures. This case bears some resemblance to the case of conformal structures, although here the action of $\mathrm{Diff}(T^2)$ on the set of structures $\mathfrak{S}_c(T^2, \mathbb{A}^2)$ is not proper. However, the action of the subgroup $\mathrm{Diff}_0(T^2)$ on $\mathfrak{S}_c(T^2, \mathbb{A}^2)$ *is* proper.

**Example 3.31** (Action of $\mathrm{Diff}(T^2)$ on $\mathfrak{S}_c(T^2, \mathbb{A}^2)$). Observe first that every complete flat affine structure on $T^2$ is homogeneous and the identity component of its automorphism group acts simply transitively. This follows from the classification given in Theorem 5.1. Therefore, like in the case of Euclidean structures, $\mathrm{Diff}_0(T^2, x_0)$ acts freely on $\mathfrak{S}_c(T^2, \mathbb{A}^2)$ and

$$\mathfrak{D}_c(T^2, \mathbb{A}^2) = \mathfrak{S}_c(T^2, \mathbb{A}^2)/\mathrm{Diff}_0(T^2, x_0) \ .$$

Since $\mathsf{hol} : \mathrm{Diff}_0(M, x_0) \backslash \mathrm{Dev}_c(T^2, \mathbb{A}^2) \to \mathrm{Hom}_c(\mathbb{Z}^2, \mathrm{Aff}(2))$ locally admits continuous equivariant sections (see the discussion before Theorem 3.25), it follows that the map

$$\mathfrak{S}_c(T^2, \mathbb{A}^2) \to \mathfrak{D}_c(T^2, \mathbb{A}^2)$$

is a locally trivial principal bundle for $\mathrm{Diff}_0(T^2, x_0)$. (It is also a universal bundle, since $\mathfrak{S}_c(T^2, \mathbb{A}^2)$ is contractible, as we see in Proposition 3.32 below.) This already implies that $\mathrm{Diff}_0(T^2, x_0)$ acts properly on $\mathfrak{S}_c(T^2, \mathbb{A}^2)$. On the other hand, $\mathrm{Diff}(T^2, x_0)$ does *not* act properly, since the action of the (extended) mapping class group of the two-torus

$$\mathrm{Diff}(T^2, x_0)/\mathrm{Diff}_0(T^2, x_0) \cong \mathrm{GL}(2, \mathbb{Z})$$

on the deformation space $\mathfrak{D}_c(T^2, \mathbb{A}^2) = \mathbb{R}^2$ is not properly discontinuous. (See also Example 3.26).

In the previous example a slightly stronger result holds. Indeed, by the proof of [2, Corollary 4.9] the projection map

$$\mathrm{Hom}_c(\mathbb{Z}^2, \mathrm{Aff}(2)) \longrightarrow \mathfrak{D}_c(T^2, \mathbb{A}^2)$$

admits a global (continuous) section. Since the space $\mathfrak{D}_c(T^2, \mathbb{A}^2) = \mathbb{R}^2$ is contractible, we may use the covering homotopy theorem to conclude that there exists a continuous section $s : \mathfrak{D}_c(T^2, \mathbb{A}^2) \to \mathfrak{S}(T^2, \mathbb{A}^2)$. This shows that the above principal bundle is trivial. (For an explicit construction of such a section, refer to Section 6.2.2 of this chapter.)

A typical application is:

**Proposition 3.32.** *The space $\mathfrak{S}_c(T^2, \mathbb{A}^2)$ of complete flat affine structures on the two-torus is contractible.*

*Proof.* The group $\mathrm{Diff}_0(T^2, x_0)$ acts freely on the set of complete flat affine structures. By the above, invariant sections exists for this action of $\mathrm{Diff}_0(T^2, x_0)$. It follows that there is a homeomorphism

$$\mathfrak{S}_c(T^2, \mathbb{A}^2) \approx \mathrm{Diff}_0(T^2, x_0) \times \mathfrak{D}_c(T^2, \mathbb{A}^2) \ .$$

In particular, since $\mathrm{Diff}_0(T^2, x_0)$ and $\mathfrak{D}_c(T^2, \mathbb{A}^2)$ are contractible, the space $\mathfrak{S}_c(T^2, \mathbb{A}^2)$ is contractible.                                                    $\square$

See Section 6.1 for a description of $\mathfrak{S}(T^2, \mathbb{A}^2)$ and $\mathfrak{S}_c(T^2, \mathbb{A}^2)$ as subsets of the affine space of torsion free flat affine connections of $T^2$.

## 3.4 Spaces of marked structures

Let $M_0$ be a fixed smooth manifold. A diffeomorphism $f : M_0 \to M$, where $M$ is an $(X, G)$-manifold is called a *marking* of $M$. Two marked $(X, G)$-manifolds $(f, M)$ and $(f', M')$ are called equivalent if there exists an $(X, G)$-equivalence $g : M' \to M$ such that $g \circ f'$ is *homotopic* to $f$. Let $\mathfrak{DM}(M_0, X, G)$ denote the set of classes of marked $(X, G)$-manifolds. By composing with $f$, every local $(X, G)$-chart for the marked manifold $(f, M)$ extends to the development map

of an $(X, G)$-structure on $M_o$. This correspondence descends to a bijection of $\mathfrak{DM}(M_0, X, G)$ with the deformation space $\mathfrak{D}(M_0, X, G)$. We can thus topologize the space of classes of markings with the topology induced from the $C^\infty$-topology on development maps.

**Example 3.33** (Teichmüller metric on $\mathfrak{T}_g$). Classically, Teichmüller space $\mathfrak{T}_g$ is represented as a space of marked conformal structures on Riemann surfaces. Let $S_0$ be a closed Riemann surface of genus $g$. A marking of a Riemann surface $R$ is an orientation preserving quasi-conformal homeomorphism $f : S_0 \to R$. Two marked surfaces $(f, R)$ and $(f', R')$ are equivalent if there exists a biholomorphic map $h : R' \to R$ such that $h \circ f'$ is *homotopic* to $f$. Teichmüller space $\mathfrak{T}_g$ is the set of classes of marked surfaces. The infimum of dilatations $K(\ell)$, where $\ell : R \to R'$ is a quasiconformal map homotopic to $f' \circ f$, defines the *Teichmüller distance* of $(f, R)$ and $(f', R')$ in $\mathfrak{T}_g$:

$$d_{\mathfrak{T}}([f, R], [f', R']) = \inf \log K(\ell) \ .$$

With the metric topology induced by $d_{\mathfrak{T}}$, $\mathfrak{T}_g$ is homeomorphic to the Fricke space

$$\mathfrak{F}(S_0) = \mathrm{Hom}_c(\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))/\mathrm{PSL}(2, \mathbb{R}),$$

as defined in Example 3.21. See [1] and [63] in Vol. I of this handbook for reference on this material. Therefore, the topology defined by Teichmüller's metric coincides with the topology on $\mathfrak{T}_g$, which is defined by the convergence of development maps for hyperbolic structures.

We may also consider various refined versions of classes of marked $(X, G)$-manifolds and corresponding deformation spaces.

**$(X, G)$-manifolds with basepoint**   Fix a basepoint $x_0 \in X$ and write $X = G/H$, where $H = G_{x_0}$. The space $\mathfrak{DM}_{\mathsf{p}}(M_0, X, G)$ of *basepointed* marked structures is defined as follows. Let $\mathsf{p} : (\tilde{M}_0, \tilde{m}_0) \to (M_0, m_0)$ be a fixed universal cover. A marking of $(M, m)$ is a based diffeomorphism $f : (M_0, m_0) \to (M, m)$. Two marked basepointed $(X, G)$-manifolds are equivalent if there exists an $(X, G)$-equivalence $g : (M', m') \to (M, m)$ such that $g \circ f'$ is homotopic to $f$ by a basepoint preserving homotopy. Let $\mathrm{Dev}_{\mathsf{p}}(M_0)$ be the set of basepoint preserving development maps. For every based local $(X, G)$ chart $\varphi$, defined near $m_0$, there exists a unique development map $D$ for $M_0$, which extends $\varphi \circ f \circ \mathsf{p}$ from a neighborhood of $\tilde{m}_0$. This correspondence induces a homeomorphism

$$\mathfrak{DM}_{\mathsf{p}}(M_0, X, G) \ \xrightarrow{\approx} \ \mathrm{Diff}_1(M_0, m_0)\backslash \mathrm{Dev}_{\mathsf{p}}(M_0)/H \ .$$

**Example 3.34** (Homogeneous $(X, G)$-structures). Note that the natural (forgetful) map $\mathfrak{DM}_{\mathsf{p}}(M, X, G) \to \mathfrak{DM}(M, X, G)$ is surjective, but it is usually

not injective. In fact, let $D$ be a development map. Then for the classes $G \circ D \circ \mathrm{Diff}_1(M, m)$ and $G \circ D \circ \mathrm{Diff}_1(M)$ to coincide it is necessary that the group of $(X, G)$-equivalences acts transitively on $M$. That is, $D$ is the development map of a homogeneous $(X, G)$-structure. We let $\mathrm{Dev}_h(M, X, G)$ denote the set of development maps of homogeneous $(X, G)$-structures.

Note further that, in general, $\mathrm{Diff}_1(M_0, m_0)$ is a proper subgroup of all basepoint preserving diffeomorphisms which are *freely* homotopic to the identity. The difference is obtained by the natural action of $\pi_1(M_0, m_0)$ on based homotopy classes of maps. However, if $\pi_1(M_0, m_0)$ is abelian, the inclusion is an isomorphism.

**Example 3.35.** Let $\mathfrak{D}_h(T^2, \mathbb{A}^2)$ be the deformation space of homogeneous flat affine structures on the two-torus. Then

$$\mathfrak{D}_h(T^2, \mathbb{A}^2) = \mathrm{Diff}_1(T^2, x_0) \backslash \mathrm{Dev}_h(T^2) / \mathrm{Aff}(2).$$

In particular, since every complete affine two-torus is homogeneous, $\mathfrak{D}_h(T^2, \mathbb{A}^2)$ contains the subspace of complete flat affine structures

$$\mathfrak{D}_c(T^2, \mathbb{A}^2) = \mathrm{Diff}_1(T^2, x_0) \backslash \mathrm{Dev}_c(T^2) / \mathrm{Aff}(2) .$$

**3.4.1 Framed $(X, G)$-manifolds** The holonomy of a marked $(X, G)$-manifold is a $G$-conjugacy class of homomorphisms. To get rid of the dependence on the conjugacy class, one introduces framed structures. The holonomy theorem implies that the deformation space of framed structures is a locally compact Hausdorff space. We shall discuss only the particular simple case of $(\mathbb{A}^n, \mathrm{Aff}(n))$-manifolds.

**Example 3.36.** Let $m \in M$. A frame $\mathcal{F}_m$ for a flat affine structure on $M$ is a choice of basis of the tangent space $T_m M$. The pair $(M, \mathcal{F}_m)$ is called a *framed flat affine manifold*. Fix a frame $\mathcal{F}_{m_0}$ for $M_0$, as well, and call a frame preserving diffeomorphism $(M_0, \mathcal{F}_{m_0}) \to (M, \mathcal{F}_m)$ a marking of $(M, \mathcal{F}_m)$. Two marked framed flat affine manifolds $(f, M, \mathcal{F}_m)$ and $(f', M', \mathcal{F}'_{m'})$ are called equivalent if there exists a frame preserving affine diffeomorphism $g : (M', \mathcal{F}'_{m'}) \to (M, \mathcal{F}_m)$ such that $g \circ f'$ is based homotopic to $f$. The set of classes is denoted $\mathfrak{D}\mathfrak{M}_f(M_0, \mathbb{A}^2)$.

Let us fix a base frame $\mathcal{E}_{x_0}$ on affine space $\mathbb{A}^n$. Given a marked framed flat affine manifold $(f, M, \mathcal{F}_m)$, there exists a unique frame preserving affine chart for $M$, which is defined near $m$. This chart lifts to a unique development map $D : (\tilde{M}_0, \tilde{\mathcal{F}}_{\tilde{m}_0}) \to (\mathbb{A}^n, \mathcal{E}_{x_0})$. The correspondence descends to a bijection

$$\mathfrak{D}\mathfrak{M}_f(M_0, \mathbb{A}^n) = \mathrm{Diff}_{1,f}(M_0, \mathcal{F}_{m_0}) \backslash \mathrm{Dev}_f(M_0) ,$$

where $\mathrm{Diff}_{1,f}(M_0, \mathcal{F}_{m_0})$ denotes the group of frame preserving diffeomorphisms which are based homotopic to the identity. By the deformation theorem, there is a map

$$\mathsf{hol} : \mathfrak{DM}_f(M_0, \mathbb{A}^n) \to \mathrm{Hom}(\pi_1(M, m_0), \mathrm{Aff}(n))$$

which is continuous and which is a local homeomorphism onto its image. *This shows that $\mathfrak{DM}_f(M_0, \mathbb{A}^n)$ is a locally compact Hausdorff space.*

As is apparent from Example 3.24, the natural map $\mathfrak{DM}_f(M_0, \mathbb{A}^n) \to \mathfrak{DM}(M_0, \mathbb{A}^n)$ is surjective, and it factors over $\mathfrak{DM}_\mathsf{p}(M_0, \mathbb{A}^n)$. The following tower of maps thus sheds some light on the topology of the deformation space of flat affine structures:

$$\mathfrak{DM}_f(M_0, \mathbb{A}^n) \qquad\qquad (3.10)$$

$$\downarrow$$

$$\mathfrak{DM}_\mathsf{p}(M_0, \mathbb{A}^n)$$

$$\downarrow$$

$$\mathfrak{D}(M_0, \mathbb{A}^n) = \mathfrak{DM}(M_0, \mathbb{A}^n) \ .$$

Note that the group $\mathrm{GL}^+(n, \mathbb{R}) = \mathrm{Diff}_1(M_0, m_0)/\mathrm{Diff}_{1,f}(M_0, \mathcal{F}_{m_0})$ acts on $\mathfrak{DM}_f(M_0, \mathbb{A}^n)$, such that the first projection map in the tower is the quotient map for this action. In particular, $\mathfrak{DM}_\mathsf{p}(M_0, \mathbb{A}^n)$ *arises as the quotient space of a locally compact Hausdorff space by a reductive group action.* (Note also that the holonomy map is equivariant with respect to the conjugation action of $\mathrm{GL}^+(n, \mathbb{R})$ on holonomy homomorphisms.)

**Example 3.37** (Homogeneous framed flat affine structures on the torus). As we have seen already in Example 3.35 above, the lower map in the tower (3.10) is a bijection on homogeneous flat affine structures, that is, $\mathfrak{D}_h(T^2, \mathbb{A}^2) = \mathfrak{DM}_{\mathsf{p},h}(T^2, \mathbb{A}^2)$. The deformation space of homogeneous structures is thus obtained as a quotient by an action of $\mathrm{GL}^+(2, \mathbb{R})$:

$$\mathfrak{D}_h(T^2, \mathbb{A}^2) = \mathfrak{DM}_{f,h}(T^2, \mathbb{A}^2)/\mathrm{GL}^+(2, \mathbb{R}) \ .$$

We shall further study this quotient space in Section 6.2. Observe that the action of $\mathrm{GL}^+(2, \mathbb{R})$ is not free, since, in fact, the Hopf tori have non-trivial stabilizers. On the other hand, as follows from the discussion in Example 3.26, $\mathrm{GL}^+(2, \mathbb{R})$ acts freely on the subspace of complete affine structures, and the map $\mathfrak{DM}_{f,c}(T^2, \mathbb{A}^2) \to \mathfrak{D}_c(T^2, \mathbb{A}^2)$ is a trivial $\mathrm{GL}^+(2, \mathbb{R})$-principal bundle.

# 4 Construction of flat affine surfaces

A flat affine manifold is called *homogeneous* if its group of affine automorphisms acts transitively.    Homogeneous flat affine manifolds may be constructed from étale affine representations of two-dimensional Lie groups in a straightforward way. Compact examples can be derived from étale affine representations of the two-dimensional group manifold $\mathbb{R}^2$ by taking quotients with a discrete uniform subgroup. Every flat affine surface constructed in this way is then a *homogeneous* flat affine torus. In fact, an easy argument (see Section 5) shows that *all* homogeneous flat affine surfaces are obtained in this way. Therefore, all homogeneous flat affine tori are affinely diffeomorphic to quotients of abelian Lie groups with left-invariant flat affine structure. This also relates homogeneous affine structures on tori to two-dimensional associative algebras, a point of view which will be discussed in Section 6. In Section 4.1, we describe the classification of abelian étale affine representations on $\mathbb{A}^2$. By the above remarks, this amounts to a rough classification of homogeneous flat affine tori.

A genuinely more geometric approach is to construct flat affine surfaces by gluing patches of affine space along their boundaries. The affine version of Poincaré's fundamental polygon theorem allows to construct flat affine tori by gluing affine quadrilaterals along their sides. The flat affine two-tori thus obtained depend on the shape of the quadrilateral and also on the particular affine transformations which are used in the gluing process. This, in turn, gives natural coordinates for an open subset in the deformation space of flat affine structures on the two-torus. As it turns out, the flat affine tori which are obtained by gluing an affine quadrilateral along its sides are all homogeneous, and they form a dense subset in the deformation space of homogeneous flat affine structures on the torus. This material is explained in Section 4.2.

To construct all flat affine two-tori, it is required to glue more general objects. In the following sections Section 4.3 and Section 4.4, we discuss in detail a construction method for flat affine tori with development image $\mathbb{A}^2 - 0$, which builds on the idea of cutting flat affine surfaces into simple building blocks. Here flat affine tori are constructed by gluing several copies of half annuli in $\mathbb{A}^2 - 0$, or cutting the surface into affine cylinders. Equivalently, these tori are obtained by gluing certain strips which are situated in the universal covering space of $\mathbb{A}^2 - 0$, and which project to annuli in $\mathbb{A}^2 - 0$. In this way also *non-homogeneous* examples of flat affine tori arise.

As follows from the main classification theorem, which will be proved in Section 5, the above construction methods exhaust all flat affine two-tori.

## 4.1  Quotients of flat affine Lie groups

If a Lie group $G$ has an étale action (cf. Definition 3.9) on affine space we call it an étale affine Lie group. An étale affine Lie group carries a natural left invariant flat affine structure, and, thus, for every discrete subgroup $\Gamma \leq G$, the quotient space $\Gamma \backslash G$ inherits the structure of a flat affine manifold. If $G$ is abelian the resulting flat affine structure is homogeneous.

The following result will be established in the course of the proof of the classification theorem (see Section 5.3):

**Proposition 4.1.** *Every homogeneous flat affine two-torus is affinely diffeomorphic to a quotient of an abelian étale affine Lie group.*

Up to affine conjugacy there are six types $\mathsf{T}, \mathsf{D}, \mathsf{C}_1, \mathsf{C}_2, \mathsf{B}, \mathsf{A}$ of étale *abelian* subgroups in the affine group $\mathrm{Aff}(2)$. Both the plane and the halfplane admit two distinct simply transitive abelian affine actions $\mathsf{T}, \mathsf{D}$, and $\mathsf{C}_1, \mathsf{C}_2$ respectively.

**Example 4.2** (Affine automorphisms of development images).

(1) (**The plane** $\mathbb{A}^2$) The groups

$$\mathsf{T} = \left\{ \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} \right\} \quad \text{and} \quad \mathsf{D} = \left\{ \begin{pmatrix} 1 & v & u + \frac{1}{2}v^2 \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

are abelian groups of affine transformations which are simply transitive on the plane.

(2) (**The half space** $\mathcal{H}$) Let $\mathcal{H}$ be the half space $y > 0$. Then

$$\mathrm{Aff}(\mathcal{H}) = \left\{ \begin{pmatrix} \alpha & z & v \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; \alpha \neq 0,\, \beta > 0 \right\}$$

is its affine automorphism group. The subgroups

$$\mathsf{C}_1 = \left\{ \begin{pmatrix} \exp(t) & z \\ 0 & \exp(t) \end{pmatrix} \right\} \subset \mathrm{GL}(2, \mathbb{R})$$

and

$$\mathsf{C}_2 = \left\{ \begin{pmatrix} 1 & 0 & v \\ 0 & \exp(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset \mathrm{Aff}(2)$$

are simply transitive abelian groups of affine transformations on $\mathcal{H}$. The half spaces $(x, y)$, $y > 0$ and $y < 0$ are open orbits for the groups $\mathsf{C}_i$.

(3) (***The sector*** $\mathcal{Q}$) Let $\mathcal{Q}$ denote the upper right open quadrant. Then

$$\mathsf{B} = \mathrm{Aff}(\mathcal{Q})^0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \,\Big|\, a > 0, b > 0 \right\} \subseteq \mathrm{GL}(2, \mathbb{R})$$

is an abelian, simply transitive *linear* group of transformations of $\mathcal{Q}$.

(4) (***The punctured plane*** $\mathbb{A}^2 - 0$)

$$\mathsf{A} = \left\{ \exp(t) \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\} \subseteq \mathrm{GL}(2, \mathbb{R}) = \mathrm{Aff}(\mathbb{A}^2 - 0)$$

is an abelian *linear* group, which is simply transitive on $\mathbb{A}^2 - 0$.

   Let $\mathsf{G} \leq \mathrm{Aff}(2)$ be one of the above groups and $\Gamma \leq \mathsf{G}$ a lattice. Then $\Gamma$ acts properly discontinuously and with compact quotient on every open orbit $U \subset \mathbb{A}^2$ of $\mathsf{G}$ and the quotient space

$$M = {}_{\Gamma} \backslash U$$

is a flat affine two-torus. The group $\Gamma$ is a discrete abelian subgroup of $\mathrm{Aff}(2)$ and, by choosing an appropriate fundamental domain, its action defines a tessellation of the open domain $U$. In fact, a convex affine quadrilateral may be chosen as fundamental domain. See Figure 10 and Figure 11 for some examples. Since $\mathsf{G}$ is abelian and centralizes $\Gamma$, the affine action of $\mathsf{G}$ on $U$ descends to $M$. Thus, $\mathsf{G}$ acts on $M$ by affine transformations, and $M$ is a homogeneous flat affine manifold.
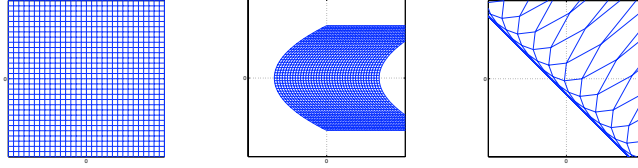


Figure 10. Tesselations of homogeneous affine domains of type $\mathsf{T}, \mathsf{D}, \mathsf{C}_1$.

   For further reference we note the following:

**Lemma 4.3** (Normalisers of étale affine groups).   (1) *The étale affine groups* $\mathsf{A}$, $\mathsf{B}$ *have index two and eight in their normalizers in* $\mathrm{Aff}(2)$. *The quotients are generated by the reflections*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R}), \textit{ respectively, } \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(2) *The normalizers in* $\mathrm{Aff}(2)$ *of the étale affine groups* $\mathsf{C}_1$, $\mathsf{C}_2$ *are*

$$\left\{ \begin{pmatrix} \alpha & z \\ 0 & \beta \end{pmatrix} \right\} \subset \mathrm{GL}(2,\mathbb{R}), \quad \left\{ \begin{pmatrix} \alpha & 0 & v \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset \mathrm{Aff}(2)$$

*respectively.*

(3) *The normalizer in* $\mathrm{Aff}(2)$ *of the étale affine group* $\mathsf{D}$ *is the semi-direct product generated by* $\mathsf{D}$ *and the linear group*

$$\mathcal{N}_{\mathsf{D}} = \left\{ \begin{pmatrix} d^2 & b & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; d \in \mathbb{R}^*,\, b \in \mathbb{R} \right\}.$$
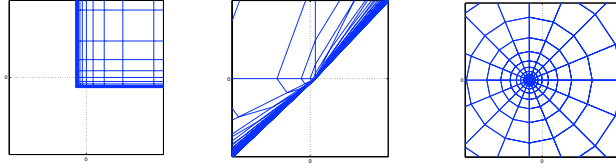


Figure 11. Tesselations of homogeneous affine domains of type $\mathsf{B}, \mathsf{C}_1, \mathsf{A}$.

In the case of the group $\mathsf{A}$, which is not simply connected, we can consider, more generally, the universal covering group $\tilde{\mathsf{A}}$ of $\mathsf{A}$. The covering homomorphism $\tilde{\mathsf{A}} \to \mathsf{A}$ turns $\tilde{\mathsf{A}}$ into an étale affine Lie group. Let $\Gamma$ be a lattice in $\tilde{\mathsf{A}}$. Then

$$M = {}_\Gamma \backslash \tilde{\mathsf{A}}$$

inherits a flat affine structure, for which the orbit map $\tilde{\mathsf{A}} \to \mathbb{A}^2 - 0$ at any point $x \in \mathbb{A}^2 - 0$ is a development map. The holonomy group $h(\Gamma) \le \mathsf{A}$ is the image of $\Gamma$ under the covering $\tilde{\mathsf{A}} \to \mathsf{A}$. Since $\Gamma$ is central in $\tilde{\mathsf{A}}$, the group $\tilde{\mathsf{A}}$ acts on $M$ by affine transformations. In particular, as before, $M$ is a *homogeneous* flat affine two-torus.

Note that, in this case, it may also happen that the holonomy $h(\Gamma)$ is *not* discrete in $\mathrm{Aff}(2)$, see Figure 12.

If the holonomy is cyclic, as is the case for Hopf tori (Example 4.8), $M$ cannot be constructed by gluing an affine quadrilateral which is contained in the development image. However, $M$ may always be obtained by gluing a strip which is situated in $\tilde{\mathsf{A}}$, see Example 4.21.
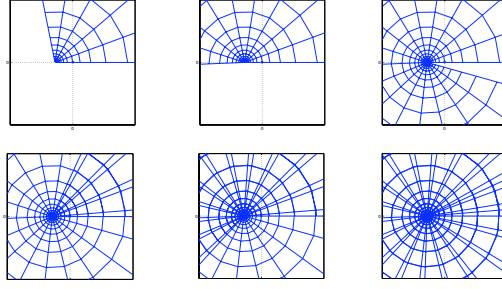
Figure 12. Development process with non-discrete holonomy group.

## 4.2  Affine gluing of polygons

Let $\mathcal{P} \subset \mathbb{A}^2$ be polygon with $\mathcal{S}$ its set of sides. Let $\{g_S \in \mathrm{Aff}(2) \mid S \in \mathcal{S}\}$ be a set of affine transformations pairing the sides of $\mathcal{P}$ and let $M$ be the corresponding identification space of $\mathcal{P}$. We say that the *affine gluing criterion*[1] holds if, for each vertex $x \in \mathcal{P}$ with cycle of edges $S_1, \ldots, S_m$, the *cycle relation* $g_{S_1} \cdots g_{S_m} = 1$ holds, and furthermore the corners at $x$ of the polygons $g_{S_1} \cdots g_{S_i} \mathcal{P}$, $i = 1, \ldots, m$, add up subsequently to a disc, while intersecting only in their consecutive boundaries. This disc then provides an affine coordinate neighborhood in the identification space of $\mathcal{P}$ defined by the pairing of sides. If the gluing criterion is satisfied the identification space $M$ inherits the structure of a flat affine manifold from $\mathcal{P}$.

The following result is the analogue of Poincaré's fundamental polygon theorem (cf. [54] for the classical version) for gluing flat affine surfaces:

**Proposition 4.4** (see [2, Proposition 2.1]). *If the affine gluing criterion holds then the group $\Gamma \subset \mathrm{Aff}(2)$ generated by the side-pairing transformations $\{g_S \mid S \in \mathcal{S}\}$ acts properly discontinuously and with fundamental domain $\mathcal{P}$ on a flat affine surface $\bar{X}$ which develops $\Gamma$-equivariantly onto an open set $U$ in $\mathbb{A}^2$. The inclusion of $\mathcal{P}$ into $\bar{X}$ identifies $M$ and the orbit space $_\Gamma \backslash \bar{X}$.*

It follows that $M$ inherits a natural flat affine structure from $\mathcal{P}$. In fact, the surface $\bar{X}$ is the holonomy covering space of $M$ and the group $\Gamma$ is the holonomy group of $M$. Note also that the construction of $\bar{X}$ is sketched in the proof of Theorem 3.15. The situation is pictured in the following commutative

---

[1]See [2] for more details on this definition

diagram of maps:

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\bar{D}} & U \subset \mathbb{A}^2 \\ \downarrow & & \downarrow \\ M = \Gamma\backslash\bar{X} & \longrightarrow & \Gamma\backslash U \ . \end{array}$$

**Example 4.5.** Figure 13 shows how to glue a trapezium $\mathcal{T}$ with angle $\alpha < \pi$. The sides $S_1$ and $S_3$ are glued with a homothety. $S_2$ and $S_4$ are glued with a rotation of angle $\alpha$. The developing image is $U = \mathbb{A}^2 - 0$. It is tessellated by the translates of $\mathcal{T}$ if and only if $m\alpha = 2\pi$ for an integer $m$. If the angle $\alpha$ is rational, $p\alpha = q2\pi$, the development $\bar{X} \to \mathbb{A}^2 - 0$ is a finite cyclic covering of degree $q$. Otherwise the development is an infinite cyclic covering and $\bar{X}$ is simply connected.
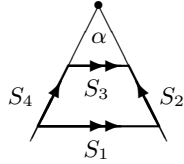


Figure 13. Gluing a trapezium in $\mathbb{A}^2$.

**4.2.1 Gluing affine quadrilaterals** Theorem 2.1 implies that the gluing criterion imposes strong restrictions on the possible combinatorial types of polygons and pairings. However, flat affine two-tori are easily obtained by gluing an affine quadrilateral $\mathcal{P}$ in the way indicated in Figure 14. The equivalence class of the flat affine manifold thus obtained depends on the affine equivalence class of $\mathcal{P}$ and the particular side pairing transformations chosen, see [2, Section 3].
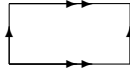


Figure 14. Gluing a torus.

The gluing conditions for such a pairing are easily verified:

**Lemma 4.6** ([2, Lemma 3.2]). *Let* $A, B \in \mathrm{Aff}(2)$, $A(0,0) = (1,0)$, $A(0,1) = p$, $B(0,0) = (0,1)$, $B(1,0) = p$. *The side pairing transformations* $\{A, A^{-1}, B, B^{-1}\}$

*for the polygon with vertices $\mathcal{P} = ((0,0), (1,0), p, (0,1))$ satisfy the gluing conditions if and only if $\det l(A) > 0$ and $\det l(B) > 0$ (where $l$ denotes the linear part of an affine transformation) and $[A, B] = \mathrm{Id}$.*

We remark further:

**Proposition 4.7.** *Every flat affine two-torus obtained by gluing a quadrilateral $\mathcal{P}$ on its sides is homogeneous. Conversely, if $M$ is a homogeneous flat affine two-torus with non-cyclic affine holonomy group, then $M$ may be obtained by gluing a quadrilateral.*

*Proof.* For the proof that the gluing torus of $\mathcal{P}$ is homogeneous, we have to appeal to some of the facts which are explained in Section 5. In particular, we use Proposition 5.2 and Proposition 5.3. The gluing conditions imply that the minimal connected abelian subgroup $N$ which contains the holonomy of $M$ is at least two-dimensional. Therefore, $N$ is one of the two-dimensional abelian subgroups listed in Example 4.2. By Proposition 5.3, $N$ acts on the development image of $M$. Let $U$ be an open orbit for $N$, which is one of the domains of Example 4.2. Then, by convexity of the homogeneous domain $U$, the polygon $\mathcal{P}$ is contained in $U$. The construction of $M$ and its development process show that the development image of $M$ is covered by the holonomy translates of $\mathcal{P}$. Since the holonomy is contained in $N$, it follows that the development image of $M$ is contained in the open orbit $U$. On the other hand, by Proposition 5.3, the development image contains $U$. This implies that the development image of $M$ equals the open orbit $U$. Therefore, $N$ acts transitively on the development image, and thus also on $M$. In particular, it follows that $M$ is homogeneous.

We omit the proof of the converse statement. A special case is treated in [2, Section 4.6]. □

**4.2.2 The gluing variety** By Lemma 4.6, the set of side pairings

$$\mathcal{V} = \{(p, A, B)\} \subset \mathbb{R}^2 \times \mathbb{R}^6 \times \mathbb{R}^6 \,,$$

which satisfy the gluing conditions for the (convex) quadrilateral

$$\mathcal{P} = (\, (0,0), (1,0), p, (0,1)\, )$$

form a semi-algebraic subset of $\mathbb{R}^2 \times \mathbb{R}^6 \times \mathbb{R}^6$. It is easily computed that $\mathcal{V}$ is four-dimensional and that the set of solutions with respect to a fixed $\mathcal{P}$ is of dimension two. We let $\mathsf{p} : \mathcal{V} \to \mathbb{R}^2$ denote the projection to the first factor. Note that the projection of $\mathcal{V}$ to the matrix factors $\mathbb{R}^6 \times \mathbb{R}^6$ defines an *embedding* $\mathcal{V} \hookrightarrow \mathrm{Hom}(\mathbb{Z}^2, \mathrm{Aff}(2))$ as a subset of the holonomy image. We call the set $\mathcal{V}$ the *gluing variety of quadrilaterals.*

**Embedding into the deformation space**    Observe that the gluing of a quadrilateral $\mathcal{P}$ naturally constructs a *framed affine two-torus*. If we choose a diffeomorphism of the unit-square with $\mathcal{P}$, the development process of the gluing extends this diffeomorphism to the development map of a *marked* framed affine two-torus (see Example 3.36). This defines a continuous map

$$\mathcal{V} \supset \mathsf{p}^{-1}(\mathcal{P}) \to \mathfrak{DM}_f(T^2, \mathbb{A}^2) \ ,$$

where $\mathsf{p} : \mathcal{V} \to \mathbb{R}^2$ is the projection to the first factor. We may furthermore choose a natural identification of the unit square with $\mathcal{P}$ (for example, by decomposing any quadrilateral into two triangles and using affine identifications of the triangles). Then, using (3.10), we obtain a continuous (open) embedding

$$\nu : \mathcal{V} \to \mathfrak{DM}_f(T^2, \mathbb{A}^2)$$

to the space of classes of framed flat affine tori. Note that $\nu$ is a section of the holonomy map $\mathsf{hol} : \mathfrak{DM}_f(T^2, \mathbb{A}^2) \to \mathrm{Hom}(\mathbb{Z}^2, \mathrm{Aff}(2))$. Since $\mathcal{V}$ also defines a slice for the $\mathrm{GL}(2, \mathbb{R})$-orbits on $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{Aff}(2))$, the map $\nu$ descends to an embedding $\mathcal{V} \to \mathfrak{DM}_{\mathsf{p}}(T^2, \mathbb{A}^2)$, whose image consists of homogeneous structures. Therefore, by the discussion in Example 3.37, *the gluing variety $\mathcal{V}$ embeds as a locally closed subset of the deformation space* $\mathfrak{D}(T^2, \mathbb{A}^2)$.

## 4.3  Tori with development image $\mathbb{A}^2 - 0$

Here we discuss how to glue flat affine tori from annuli which are contained in the once punctured plane or the universal covering flat affine manifold of the once punctured plane.

**4.3.1  Hopf tori and quotients of $\mathbb{A}^2 - 0$**  The simplest examples of flat affine tori with development image $\mathbb{A}^2 - 0$ are obtained by gluing closed annuli along their boundary curves.

**Example 4.8** (Hopf tori). Let $A_\lambda$, $\lambda > 0$, be a dilation with scaling factor $\lambda$, and $\Gamma = \langle A_\lambda \rangle$ the subgroup of $\mathrm{GL}(2, \mathbb{R})$ generated by $A_\lambda$. Then $\Gamma$ acts properly discontinuously on $\mathbb{A}^2 - 0$ and the quotient space

$$\mathcal{H}_\lambda = \ _\Gamma \backslash \ (\mathbb{A}^2 - 0)$$

is a compact flat affine two-torus $\mathcal{H}_\lambda$, which is called a *Hopf torus*.

The Hopf torus $\mathcal{H}_\lambda$ is obtained by gluing a closed annulus $\mathcal{A}_\lambda \subset \mathbb{A}^2 - 0$ of width $\lambda$ along its boundary circles, see Figure 15.

The geometric construction of the Hopf tori $\mathcal{H}_\lambda$ may be refined as follows:
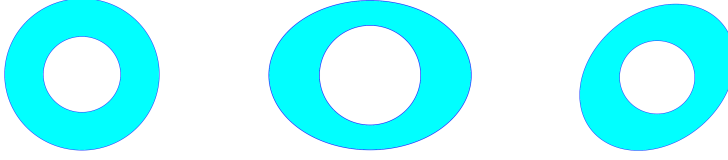
Figure 15. Gluing of annuli: Hopf torus, non-homogeneous flat affine tori.

**Example 4.9** (Finite coverings of Hopf tori). Let

$$X_k \longrightarrow \mathbb{A}^2 - 0$$

be a $k$-fold covering flat affine manifold. Then we may lift the action of $A_\lambda$ on $\mathbb{A}^2 - 0$ to a properly discontinuous action of an affine transformation $A_{\lambda,k}$ of $X_k$. The quotient spaces

$$\mathcal{H}_{\lambda,k} = \langle A_{\lambda,k} \rangle \backslash X_k$$

are flat affine manifolds, which are $k$-fold covering spaces of $\mathcal{H}_\lambda$. Geometrically, $X_k$ is a topological annulus with a flat affine structure which is obtained by cutting $\mathbb{A}^2 - 0$ at a radial line and then gluing $k$ copies of $\mathbb{A}^2 - 0$ along this geodesic ray. A *geodesic* in a flat affine manifold is a curve which corresponds to a straight line in all affine coordinate charts. Thus, correspondingly, the manifolds $\mathcal{H}_{\lambda,k}$ are obtained by gluing $k$ copies of $\mathcal{H}_\lambda$ at a closed geodesic.

Note that the family of Hopf tori $\mathcal{H}_{\lambda,k}$ gives a simple example of a family of distinct flat affine manifolds which have identical holonomy homomorphism.

**Example 4.10** (Finite quotients of Hopf tori). Let $R_\alpha$ be a rotation with angle $\alpha = \frac{p}{q}\pi$ a rational multiple of $\pi$. Then the finite group of rotations of order $2q$ generated by $R_\alpha$ acts without fixed points on $\mathbb{A}^2 - 0$ and on the Hopf tori $\mathcal{H}_{\lambda,k}$. Therefore, the quotient spaces

$$\mathcal{H}_{\lambda,\alpha,k} = \langle R_\alpha \rangle \backslash \mathcal{H}_{\lambda,k}$$

are flat affine two-tori.

Since $A_\lambda$ is in the center of $\mathrm{GL}(2,\mathbb{R})$, $\mathcal{H}_\lambda$ is a homogeneous flat affine manifold with affine automorphism group

$$\mathrm{Aff}(\mathcal{H}_\lambda) = \mathrm{GL}(2,\mathbb{R})/\Gamma .$$

Hence, its finite coverings $\mathcal{H}_{\lambda,k}$ are homogeneous, as well. Similarly, $\mathcal{H}_{\lambda,\alpha,k}$ are homogeneous flat affine two-tori, with $\mathrm{Aff}(\mathcal{H}_{\lambda,\alpha})^0$ isomorphic to $\mathrm{GL}(1,\mathbb{C})/\Gamma$, except for $\alpha = \pi$. In the latter case $\mathrm{Aff}(\mathcal{H}_{\lambda,\pi}) = \mathrm{PGL}(2,\mathbb{R})/\Gamma$.

**Expanding holonomy**    Non-homogeneous quotients of $\mathbb{A}^2 - 0$ may be constructed by using expanding elements of $\mathrm{GL}(2,\mathbb{R})$. A matrix $A \in \mathrm{GL}(2,\mathbb{R})$ is called an *expansion* if it has real eigenvalues $\lambda_1, \lambda_2 > 1$. ($A^{-1}$ is then called a *contraction*.) Every expansion acts properly discontinuously on $\mathbb{A}^2 - 0$, see Figure 16. This motivates the following:

**Definition 4.11** (Expanding elements). A matrix $A \in \mathrm{GL}(2,\mathbb{R})$ is called *expanding* if it acts properly on $\mathbb{A}^2 - 0$ and every compact subset of $\mathbb{A}^2 - 0$ is moved to infinity by its iterates $A^k$, $k \to \infty$.

Note that, if $A$ is expanding, it is either an expansion, or a product of an expansion with $\mathrm{R}_\pi$, or it is conjugate to a product of a dilation and a rotation.

Figure 16. Dynamics of expanding elements in $\mathrm{GL}(2,\mathbb{R})$.

Figure 17. Dynamics of non-expanding elements in $\mathrm{GL}_2(\mathbb{R})$.

**Example 4.12** (Tori with expanding holonomy). If $A$ is an expanding element then the quotient space

$$\mathcal{H}_A = \langle A \rangle \backslash \mathbb{A}^2 - 0$$

is a flat affine two-torus with development image $\mathbb{A}^2 - 0$. If $A$ is an expansion, the torus $\mathcal{H}_A$ is obtained by gluing an annulus $\mathcal{A}_A \subset \mathbb{A}^2 - 0$, as indicated in Figure 15. Note, if $A$ is an expansion which is not a dilation then $\mathcal{H}_A$ is a flat affine torus, which is *not* homogenous.

**4.3.2 Quotients of $\widetilde{\mathbb{A}^2 - 0}$**  We consider the universal covering flat affine manifold

$$\mathsf{q} : \widetilde{\mathbb{A}^2 - 0} \longrightarrow \mathbb{A}^2 - 0$$

of the open domain $\mathbb{A}^2 - 0$ in $\mathbb{A}^2$. Let $\mathrm{Aff}(\widetilde{\mathbb{A}^2 - 0})$ be its group of affine diffeomorphisms.

**Universal covering of** $\mathrm{GL}(2, \mathbb{R})$   The development $\mathsf{q}$ induces a surjective homomorphism

$$\mathsf{p} : \mathrm{Aff}(\widetilde{\mathbb{A}^2 - 0}) \longrightarrow \mathrm{Aff}(\mathbb{A}^2 - 0) = \mathrm{GL}(2, \mathbb{R}) ,$$

which exhibits $\mathrm{Aff}(\widetilde{\mathbb{A}^2 - 0})$ as the universal covering group of $\mathrm{GL}(2, \mathbb{R})$.

Let $\mathrm{R}_\pi \in \mathrm{GL}(2, \mathbb{R})$ denote rotation by $\pi$. The center of the group

$$\mathrm{Aff}(\widetilde{\mathbb{A}^2 - 0}) = \widetilde{\mathrm{GL}}(2, \mathbb{R})$$

is therefore generated by an element $\tau$ which satisfies $\mathsf{p}(\tau) = \mathrm{R}_\pi$, and the kernel of $\mathsf{p}$ is generated by $\tau^2$. (cf. Section A.)

**Polar coordinates.**   We let $(r, \theta)$ denote polar coordinates for $\widetilde{\mathbb{A}^2 - 0}$. Then

$$\tau : (r, \theta) \mapsto (r, \theta + \pi) .$$

More generally, the universal covering $\widetilde{\mathrm{SO}}(2, \mathbb{R})$ of the rotation group is a subgroup of $\widetilde{\mathrm{GL}}(2, \mathbb{R})$ which acts by translations in the $\theta$-direction.

**Elements with non-zero rotation angle.**   Let $B \in \mathrm{GL}^+(2, \mathbb{R})$ have positive eigenvalues. After conjugation, we may assume that $B$ preserves the horizontal coordinate axis in $\mathbb{A}^2$. We let $\tilde{B} = \tilde{B}_0$ denote the lift of $B$ to $\widetilde{\mathbb{A}^2 - 0}$ which preserves the line $\theta = 0$. It follows that $\tilde{B}$ preserves all horizontal strips

$$\bar{\Omega}_\ell = \left\{ (r, \theta) \mid \ell\, \pi \leq \theta \leq (\ell + 1)\, \pi \right\}$$

and their boundary components. We observe that (the group generated by) any other lift

$$\tilde{B}_k = \tau^k \tilde{B} , \; k \neq 0 ,$$

acts properly on $\widetilde{\mathbb{A}^2 - 0}$.

**Definition 4.13** (Non-zero angle of rotation). Let $\tilde{B} \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$. We say that $\tilde{B}$ has a non-zero rotation angle if $\tilde{B}$ acts properly on $\widetilde{\mathbb{A}^2 - 0}$, and for every compact subset the $\theta$ coordinates are unbounded under the iterates $\tilde{B}^k$, $k \to \infty$.

The property to have non-zero rotation is an affine invariant that is, it is invariant by conjugation in $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$. In particular, $\tilde{B} \neq 1$ has non-zero angle of rotation if and only if $\tilde{B}$ is conjugate to an element of $\widetilde{\mathrm{SO}}(2)$ or $B$ has positive eigenvalues and $\tilde{B} = \tilde{B}_k = \tau^k \tilde{B}_0$, $k \neq 0$, as above.

**Proper actions on** $\widetilde{\mathbb{A}^2-0}$**.** Let $\bar{\mathcal{H}}_0^k = \bigcup_{\ell=0\ldots k} \bar{\Omega}_\ell$ be the successive union of $k$ strips $\bar{\Omega}_\ell$. Note that the development image of $\bar{\Omega}_\ell$ is a closed halfspace with the origin 0 removed, and the development image of $\bar{\mathcal{H}}_0^k$ is $\mathbb{A}^2-0$, $k \geq 1$ (compare also Figure 23).

**Example 4.14** (Affine cylinders without boundary). Consider the quotient flat affine manifolds

$$X_{B,k} = \langle \tilde{B}_k \rangle \backslash \widetilde{\mathbb{A}^2-0}, \; k \geq 1 .$$

These are open affine cylinders which are obtained by gluing $\bar{\mathcal{H}}_0^k$ along its two incomplete boundary geodesics. The development image of $X_{B,k}$ is $\mathbb{A}^2-0$, and its holonomy group is generated by $\mathrm{R}_\pi^k B$.

Let $A$ be an expansion which commutes with $B$ and $\tilde{A} = \tilde{A}_0$ the lift of $A$ which preserves the line $\theta = 0$. Then $\tilde{A}$ acts properly on $X_{\tilde{B}}$.

**Example 4.15** (Quotients of $\widetilde{\mathbb{A}^2-0}$). We obtain the quotient affine torus

$$\mathcal{T}_{\tilde{A},\tilde{B}} = \mathcal{T}_{A,B,k} = \langle \tilde{A} \rangle \backslash X_{B,k} , k \neq 0 .$$

The holonomy homomorphism of $\mathcal{T}_{A,B,k}$ is determined by $A$, $B$ and the parity of $k$.

**Example 4.16** (Affine cylinders without boundary, general case). Let $\tilde{B}$ be an element of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ which has non-zero rotation. Then the quotient flat affine manifolds

$$X_{\tilde{B}} = \langle \tilde{B} \rangle \backslash \widetilde{\mathbb{A}^2-0}$$

are open cylinders which are obtained by gluing a strip

$$\bar{\mathcal{H}}_\alpha = \{(r,\theta) \mid 0 \leq \theta \leq \alpha\}$$

in $\widetilde{\mathbb{A}^2-0}$ along its two boundary geodesics.

Now let $B \in \mathrm{GL}(1,\mathbb{C})$ and $\tilde{B}$ a lift with non-zero rotation, and $A \in \mathrm{GL}(1,\mathbb{C})$ an expanding element. Then $\tilde{A}$ acts properly on $X_{\tilde{B}}$ if and only if $\tilde{A}$ and $\tilde{B}$ generate a lattice in $\widetilde{\mathrm{GL}}(1,\mathbb{C})$.

**Example 4.17.** The quotient flat affine torus

$$\mathcal{T}_{\tilde{A},\tilde{B}} = \langle \tilde{A} \rangle \backslash X_{\tilde{B},k} ,$$

is a homogeneous flat affine two-torus with holonomy in $\mathrm{GL}(1,\mathbb{C})$.

## 4.4 Affine cylinders with geodesic boundary

We show that by gluing flat affine cylinders whose boundary curves are incomplete geodesics we may construct flat affine two-tori with development image $\mathbb{A}^2 - 0$. This yields another construction of the manifolds $\mathcal{T}_{A,B,k}$ which have been introduced in Example 4.15. As a matter of fact, as a key step in the course of the proof of Theorem 5.1, we shall show that all *non-homogeneous* flat affine tori may be obtained in this way.

**Example 4.18** (Affine cylinders with geodesic boundary). Let $\bar{\mathcal{H}}_0$ be the closed upper halfplane with the origin 0 removed. If $A$ is an expansion then

$$\mathcal{C}_A = \langle A \rangle \backslash \bar{\mathcal{H}}_0$$

is topologically an annulus with two boundary components, and it is also a flat affine manifold with (incomplete) geodesic boundary, see Figure 18 and Figure 19. We call $\mathcal{C}_A$ an affine cylinder.[2]
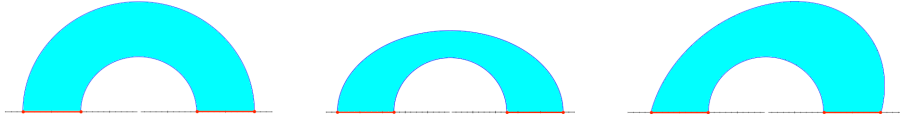


Figure 18. Gluing of flat affine cylinders $\mathcal{C}_A$.

More generally, let $\bar{\mathcal{H}}_0^k = \bigcup_{\ell=0\ldots k} \bar{\Omega}_\ell$, and $\tilde{A}$ the lift of $A$, which preserves the line $\theta = 0$. Then the manifold with boundary $\mathcal{C}_A^k = \langle \tilde{A} \rangle \backslash \bar{\mathcal{H}}_0^k$ is called an *affine cylinder*.



Figure 19. Affine cylinder $\mathcal{C}_A^3$ with geodesic boundary.

**4.4.1 Gluing of flat affine cylinders** Let $A$ be an expansion and $B \in \mathrm{GL}^+(2, \mathbb{R})$, commuting with $A$, such that $B$ has positive eigenvalues. Then, as

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[2]Benoist [12] calls $\mathcal{C}_A$ an annulus.

shown by Example 4.15, every lift $\tilde{B}_k$, $k \geq 1$ acts properly on $\langle\tilde{A}\rangle\backslash\widetilde{\mathbb{A}^2-0}$ and yields the quotient flat affine torus

$$\mathcal{T}_{A,B,k} = \langle\tilde{A}\rangle\backslash X_{B,k} \ .$$

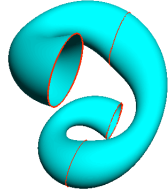Thus, geometrically, the flat affine torus $\mathcal{T}_{A,B,k}$ is constructed by gluing the flat affine cylinder $\mathcal{C}_A^k$ along its two boundary geodesics using the transformation $\tilde{B}_k$, see Figure 20.
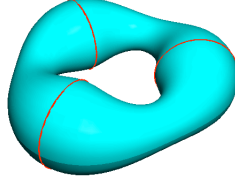


Figure 20. Gluing a flat affine torus $\mathcal{T}_{A,B,3}$.

**Remark 4.19.** Note that $M = \mathcal{T}_{A,B,k}$ is a homogeneous flat affine two-torus if and only if its holonomy group $h(\Gamma) = \langle A, B\rangle$ is contained in the group of dilations, in which case $\mathrm{Aff}(M)$ is a finite covering group of $\mathrm{PSL}(2, \mathbb{R})$, and $M$ is a Hopf torus.

**Remark 4.20.** Similarly, if $A$ is an expansion and $h(\Gamma) = \langle A, B\rangle$ is a discrete group of rank two then $h(\Gamma)$ acts properly discontinuously and with compact quotient either on $\mathcal{H}$ or on $\mathcal{Q}$. The corresponding torus is obtained by gluing $\mathcal{T}_A$ or $\mathcal{T}_{A,\alpha}$, $\alpha < \pi$.

Finally, we remark that a homogeneous flat affine torus with development image $\mathbb{A}^2 - 0$ may be constructed by gluing cylinders if and only if it admits a closed (non-complete) geodesic:

**Example 4.21** (Homogeneous tori with a closed geodesic). Let $A_\lambda$ be a dilation, and $B \in \mathrm{GL}(1, \mathbb{C})$. Then every lift of $B$ to $\widetilde{\mathbb{A}^2-0}$ is of the form

$$\tilde{B}_k : \ (r, \theta) \mapsto (\lambda r, \theta + \alpha) \ , \ \alpha = \alpha_0 + 2k\pi$$

with $\alpha_0 \in [0, 2\pi)$. If $\alpha \neq 0$, we define $X_{B,k} = \langle\tilde{B}_k\rangle\backslash\widetilde{\mathbb{A}^2-0}$, and

$$\mathcal{T}_{\lambda,B,k}^h = \langle\tilde{A}_\lambda\rangle\backslash X_{B,k} \ .$$

Let $\bar{\mathcal{H}}_\alpha = \{(r, \theta) \mid 0 \leq \theta \leq \alpha\}$ be a strip in $\widetilde{\mathbb{A}^2-0}$, and

$$\mathcal{C}_{\lambda,\alpha} = \langle\tilde{A}_\lambda\rangle\backslash\bar{\mathcal{H}}_\alpha \ .$$

Then, $\mathcal{T}^h_{\lambda,B,k}$ is a homogeneous flat affine two-torus which is obtained by gluing the cylinder $\mathcal{C}_{\lambda,\alpha}$ with $\tilde{B}_k$.
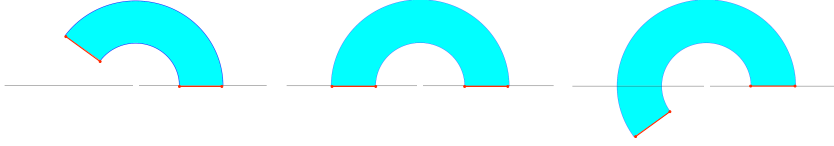


Figure 21. Affine cylinders $\mathcal{C}_{\lambda,\alpha}$, $\alpha < 2\pi$.

## 5  The classification of flat affine structures on the two-torus

The classification of flat affine structures on the two-torus was carried out by Kuiper [50] and completed by Nagano-Yagi in [61], and independently also by Furness and Arrowsmith in [28]. Later on much of the work in [61] was clarified and beautifully generalised in Benoist's paper [11]. In this section we describe the classification result in detail and explain its proof, following loosely along the lines of [61], and employing also the main ideas from [11, 12] to establish in Proposition 5.8 the crucial fact that the development map of a flat affine two-torus is always a covering map onto its image.

**Theorem 5.1.** *Let $M$ be a flat affine two-torus. Then $M$ is affinely diffeomorphic to either*

(1) *a quotient of a simply connected two-dimensional affine homogeneous domain by a properly discontinuous group of affine transformations.*

(2) *or a quotient space of the universal covering $\widetilde{\mathbb{A}^2 - 0}$ of the once punctured plane.*

*In particular, the universal covering flat affine manifold of $M$ is affinely diffeomorphic to the affine plane $\mathbb{A}^2$, the half-plane $\mathcal{H}$, or the sector (quarter plane) $\mathcal{Q}$, in the first case, and to $\widetilde{\mathbb{A}^2 - 0}$, in the second case.*

The first step in the proof of Theorem 5.1 consists of the determination of the open domains in $\mathbb{A}^2$ which appear as the development images of flat affine structures on the two-torus. This is done in Section 5.1. If $M$ is homogeneous then the development map is a covering map. The main step is then the determination of the structure of flat affine two-tori with development image

$\mathbb{A}^2 - 0$, which are not homogeneous. This is carried out in Section 5.3. We prove that such tori may be obtained by gluing affine cylinders with geodesic boundary. We deduce that also in this case the development map of $M$ is a covering map onto its image.

The following further consequences are implied by the theorem or its proof.

**Classification of divisible affine domains**     An affine domain is called *divisible* if it admits a discontinuous affine action with compact quotient. Since they admit a simply transitive abelian group, all development images of flat affine structures on the two-torus are divisible by abelian discrete groups (isomorphic to $\mathbb{Z}$ or $\mathbb{Z}^2$). Conversely, by Benzécri's theorem, every divisible plane affine domain is the development image of a flat affine structure on the two-torus. By Theorem 5.1, the universal covering of a flat affine two-torus is a homogeneous flat affine manifold, which covers a convex divisible homogeneous domain in $\mathbb{A}^2$.

**The affine automorphism group of $M$**

(1) If $M$ has development image $\mathbb{A}^2$, the half-plane $\mathcal{H}$, or the sector $\mathcal{Q}$, then $M$ is a homogeneous flat affine manifold. The connected component $\mathrm{Aff}(M)^0$ of the group $\mathrm{Aff}(M)$ of affine diffeomorphisms of $M$ is a two-dimensional compact abelian Lie group, acting transitively and freely on $M$.

(2) If the development image is the once-punctured plane $\mathbb{A}^2 - 0$ and $M$ is homogeneous, then the group $\mathrm{Aff}(M)^0$ is either a quotient of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, as is the case for Hopf tori (Examples 4.8, 4.9), or $\mathrm{Aff}(M)^0$ is a quotient of $\mathrm{GL}(1, \mathbb{C})$, as in Example 4.10. In either case, the action of $\mathrm{GL}(1, \mathbb{C})$ on $\mathbb{A}^2 - 0$ descends to a transitive and free action of a two-dimensional compact abelian Lie group on $M$.

(3) Otherwise, $\mathrm{Aff}(M)^0$ is a two-dimensional abelian connected Lie group which has a one-dimensional compact factor. In this case, $M$ is not homogeneous, as in Example 4.12.

(4) The affine automorphism group of $M$ acts prehomogeneously on $M$, that is, it has only finitely many orbits on $M$.

(5) The one-dimensional orbits of $\mathrm{Aff}(M)^0$ are non-complete geodesics in $M$ along which $M$ may be cut into flat affine cylinders.

**Homogeneous and complete flat affine tori**

(1) Every *homogeneous* flat affine two-torus $M$ is affinely diffeomorphic to a quotient of an abelian étale affine Lie group of type $\mathsf{T}$, $\mathsf{D}$, $\mathsf{C}_1$, $\mathsf{C}_2$, $\mathsf{B}$ or $\mathsf{A}$ as listed in Example 4.2.

(2) Every *complete* flat affine two-torus $M$ is affinely diffeomorphic to a quotient of an abelian simply transitive affine Lie group of type $\mathsf{T}$ or $\mathsf{D}$. In particular, $M$ is also a homogeneous flat affine two-torus.

## 5.1 Development images

The classification of development images is as follows:

**Proposition 5.2.** *Let $M$ be a flat affine two-torus. Then the development image of $M$ is either the affine plane $\mathbb{A}^2$, the half-plane $\mathcal{H}$, the sector (quarter plane) $\mathcal{Q}$ or the once-punctured plane $\mathbb{A}^2 - 0$, respectively.*

**Proof of Proposition 5.2**   Let $h(\Gamma) \leq \mathrm{Aff}(2)$ be the holonomy group of $M$. Let $N$ be the identity component of a maximal abelian subgroup of $\mathrm{Aff}(2)$ which contains $h(\Gamma)$. Note that $N$ contains the identity component of the Zariski-closure of $h(\Gamma)$. Therefore, $h(\Gamma) \cap N$ is of finite index in $h(\Gamma)$. We let $\tilde{N}$ denote the universal covering group of $N$. The first observation is that $N$ *acts on the development image*, and it has only finitely many orbits on $\mathbb{A}^2$.

**Proposition 5.3.** *The action of $N$ on $\mathbb{A}^2$ lifts via the development map to an action of $\tilde{N}$ on the universal covering flat affine manifold $\tilde{M}$ of $M$. Moreover, it follows that*

(1) *$N$ acts on the development image $\Omega$ of $M$.*

(2) *$N$ has only finitely many orbits on $\Omega$.*

*Proof.* Let $Y$ be an affine vector field on $\mathbb{R}^2$ which is tangent to the action of $N$, and let $\bar{Y}$ denote its lift to $\tilde{M}$ via $D$. Since $h(\Gamma)$ commutes with $N$, the vector field $\bar{Y}$ is $\Gamma$-invariant and projects to a vector field on $M$. Since $M$ is compact, the flow of $\bar{Y}$ is complete. Therefore, the action of $N$ integrates to an action of the universal covering group $\tilde{N}$ on $\tilde{M}$, such that $D(\tilde{n}x) = nD(x)$, where $\tilde{n} \in \tilde{N}$ and $n = h(\tilde{n})$ is its image in $N$. This implies (1).

To prove (2), we note that, up to affine conjugacy, every maximal abelian and connected subgroup $N$ of $\mathrm{Aff}(2)$ is either one of the abelian groups $\mathsf{T}$, $\mathsf{D}$, $\mathsf{C}_1, \mathsf{C}_2$, $\mathsf{B}$, $\mathsf{A}$ (as listed in Example 4.2), or the group

$$
\mathsf{N} = \left\{ \begin{pmatrix} 1 & u & v \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; u, v \in \mathbb{R} \right\} \; .
$$

All of the groups appearing in Example 4.2 are simply transitive on an affine domain, and they have finitely many orbits on $\mathbb{A}^2$. This shows (2).

To complete the proof, we contend that $h(\Gamma)$ is *not* contained in the group $\mathsf{N}$. We remark that the orbits of $\mathsf{N}$ are the horizontal lines on $\mathbb{R}^2$. If $h(\Gamma)$

is contained in $\mathsf{N}$, then by (1), $\Omega$ is a union of orbits. Thus horizontal lines define a one-dimensional foliation of $\Omega$, which is preserved by $h(\Gamma)$. This, in turn, defines a foliation on the manifold $M$, which has an open subset of the real line as its space of leaves. This is not possible, since $M$ is compact: The space of leaves is a quotient of $M$, and therefore it is a compact and closed subset of the line as well. $\qquad\square$

It follows that the development image $\Omega$ is a finite union of orbits of one of the connected abelian groups listed in Example 4.2. Since $\Omega$ is a connected open subset of $\mathbb{A}^2$, as well, it follows that $\Omega$ must be one of the domains listed in Proposition 5.2. Conversely, as follows from Section 4.1, each of these domains appears as the development image of a homogeneous flat affine structure on the torus. This completes the proof of Proposition 5.2.

## 5.2 The classification of manifolds modeled on $(\mathbb{A}^2 - 0, \mathbf{GL}(2, \mathbb{R}))$

As follows from the proof of Theorem 5.1, every compact manifold $M$ modeled on $(\mathbb{A}^2 - 0, \mathrm{GL}(2, \mathbb{R}))$ is either complete and the development image of $M$ is $\mathbb{A}^2 - 0$, or $M$ is (isomorphic to) a quotient of the open quadrant $\mathcal{Q}$, or a quotient of an open half space $\mathcal{H}$. In the first case

$$M = {}_\Gamma \backslash \widetilde{\mathbb{A}^2 - 0},$$

where $\Gamma \leq \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ is a discontinuous subgroup, and in the second case the affine holonomy group $\Gamma$ is a discrete subgroup of $\mathsf{B} \leq \mathrm{GL}(2, \mathbb{R})$, $\mathsf{C}_1 \leq \mathrm{GL}(2, \mathbb{R})$ respectively.

We arrive at the following classification theorem for $(\mathbb{A}^2 - 0, \mathrm{GL}(2, \mathbb{R}))$-manifolds which are complete (see also Corollary 5.11):

**Theorem 5.4.** *Let $M$ be a compact complete $(\mathbb{A}^2 - 0, \mathrm{GL}(2, \mathbb{R}))$-manifold. If $M$ is not homogeneous then it is isomorphic to a torus $\mathcal{T}_{\tilde{A}, \tilde{B}}$, as constructed in Section 4.4.1 and Example 4.15. Moreover, $M$ is homogeneous if and only if it can be modeled on $(\mathbb{A}^2 - 0, \mathrm{GL}(1, \mathbb{C}))$.*

In particular, if $M$ is not homogeneous then it is obtained by gluing flat affine cylinders $\mathcal{C}_A^k$, where $A$ is an expansion and $\tilde{B} \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ has non-zero angle of rotation and commutes with $A$. Furthermore if $A$ is a dilation, $B$ cannot be conjugate to an element of $\mathrm{GL}(1, \mathbb{C})$.

**Example 5.5** (Holonomy in $\mathrm{GL}^+(2, \mathbb{R})$ is not injective). This phenomenon already occurs for homogeneous flat affine manifolds which are quotients of

the universal covering flat affine Lie group $\tilde{\mathsf{A}}$ of the étale flat affine group $\mathsf{A} = \mathrm{GL}(1,\mathbb{C}) \leq \mathrm{GL}(2,\mathbb{R})$. Here different lattices $\Gamma_1$ and $\Gamma_2$ of $\mathsf{A}$ determine non-isomorphic flat affine manifolds. However, different lattices may project to the same holonomy group in $\mathsf{A}$.

More striking examples arise as a consequence of the construction of the tori $\mathcal{T}_{\tilde{A},\tilde{B}}$, as constructed in Section 4.4.1. In fact, for every non-complete homogeneous flat affine manifold modeled on $\mathsf{C}_1$ or $\mathsf{B}$ one can construct a *non-homogeneous* $(\mathbb{A}^2 - 0, \mathrm{GL}^+(2,\mathbb{R}))$-manifold which has the same holonomy group in $\mathrm{GL}^+(2,\mathbb{R})$. These examples show in particular that the affine holonomy group does *not* determine the development image.

We can consider also the corresponding $(\widetilde{\mathbb{A}^2 - 0}, \widetilde{\mathrm{GL}}^+(2,\mathbb{R}))$-manifolds. Here we have:

**Theorem 5.6.** *All compact $(\widetilde{\mathbb{A}^2 - 0}, \widetilde{\mathrm{GL}}^+(2,\mathbb{R}))$-manifolds are determined up to isomorphism by their holonomy group in $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$.*

*Proof.* For the complete manifolds the rigidity is shown in Example 3.16. In particular, complete and non-complete manifolds do not share the same holonomy in $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ (although they often do in $\mathrm{GL}^+(2,\mathbb{R})$). In fact, the holonomy group of every complete manifold has an element with non-zero angle of rotation (cf. Definition 4.13), which is not possible if the development image is one of the domains $\mathcal{H}$, $\mathcal{Q}$. Similarly the non-complete examples are lattice quotients of a simply connected abelian Lie group contained in $\mathrm{GL}^+(2,\mathbb{R})$ which acts simply transitively on some open domain in $\mathbb{A}^2$. Therefore, these manifolds are determined by their affine holonomy group. $\qquad\square$

Also the following is an immediate consequence of the above classification result:

**Corollary 5.7.** *Let $\Gamma \leq \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ be a non-finite discrete subgroup which is acting properly on $\widetilde{\mathbb{A}^2 - 0}$. Then one of the following hold:*

(1) $\Gamma$ *is isomorphic to $\mathbb{Z}$. Moreover, it is generated by an expansion, or it is generated by an element of non-zero rotation.*

(2) $\Gamma$ *is isomorphic to $\mathbb{Z}^2$ and it is conjugate to one of the subgroups constructed in Examples 4.15 or 4.17.*

## 5.3 The global model spaces

The classification theorem, Theorem 5.1, implies that there do exists four simply connected flat affine manifolds which appear as the universal covering

space of a flat affine two-torus. These simply connected model spaces for two-dimensional compact flat affine manifolds are the plane $\mathbb{A}^2$, the half-plane $\mathcal{H}$, the sector $\mathcal{Q}$, and $\widetilde{\mathbb{A}^2 - 0}$, the universal covering space of the once punctured plane.

This follows from the classification of development images once the following fact is established.

**Proposition 5.8.** *The development map of a flat affine two-torus $M$ is a covering map.*

The main step in the proof of Proposition 5.8 relies on the decomposition of $M$ into fundamental pieces, which are called bricks. This concept is due to Benoist [11]. The bricks in this case are flat affine cylinders with geodesic boundary. This brick decomposition for flat affine two-tori resembles the pants decomposition for closed hyperbolic surfaces (see [65]). *It will also serve us in the parametrisation of the deformation space in Section 6.3.2.*

**5.3.1 The brick decomposition for flat affine two-tori** Let $\tilde{M}$ be the universal covering flat affine manifold of $M$, $\Gamma \leq \mathrm{Aff}(\tilde{M})$ the group of covering transformations, and $D : \tilde{M} \to \mathbb{A}^2$ the development map. Let $N \leq \mathrm{Aff}(2)$ be the identity component of a maximal abelian subgroup containing $h(\Gamma)$, as in Section 5.1. By Proposition 5.3, the action of $N$ on the development image $D(\tilde{M})$ lifts to an action of its universal covering $\tilde{N}$ on $\tilde{M}$, such that $D$ is an equivariant map $\tilde{M} \to D(\tilde{M})$.

**Homogeneous flat affine tori** In case that the development image $D(\tilde{M})$ is $\mathbb{A}^2$, or the half-plane $\mathcal{H}$, or the sector $\mathcal{Q}$, $N$ is simply connected and acts simply transitively on $D(\tilde{M})$. It follows that $\tilde{N}$ acts simply transitively on $\tilde{M}$, and $D$ is an equivariant local diffeomorphism. Hence, $D : \tilde{M} \to D(\tilde{M})$ is an affine diffeomorphism. If the development image is $\mathbb{A}^2 - 0$, and $N$ is conjugate to $\mathrm{GL}(1, \mathbb{C})$ then $\tilde{N}$ acts simply transitively on $\tilde{M}$. It follows that $D$ is a covering map. We thus have an affine covering

$$\widetilde{\mathbb{A}^2 - 0} \stackrel{D}{\longrightarrow} \mathbb{A}^2 - 0 \ .$$

This proves that the development map is a covering map for all homogeneous flat affine tori $M$.

**Inhomogeneous flat affine tori** We assume now that $M$ is *not* homogeneous. Therefore, the development image of $M$ is $\mathbb{A}^2 - 0$ and $N$ is different from $\mathrm{GL}(1, \mathbb{C})$. Then $N$ equals either the group of diagonal matrices with positive entries $\mathsf{B}$ or the group $\mathsf{C}_1$ (compare Example 4.2). The open orbits of

$N$ on $\mathbb{A}^2 - 0$ are the open quadrant in the case $N = \mathsf{B}$, or the open half space in the case $N = \mathsf{C}_1$. In particular, in this case, $N$ does not act transitively on $\mathbb{A}^2 - 0$, and therefore $M$ is not a homogeneous flat affine torus. However, the orbits of $N$ on $M$ decompose $M$ into finitely many pieces, the *bricks*, from which $M$ is constructed.

**Proposition 5.9** (Brick Lemma for the flat affine two-torus). *Let $\Omega = \tilde{N}\tilde{x}_0$ be an open orbit of $\tilde{N}$ on $\tilde{M}$, and $\bar{\Omega}$ the closure of $\Omega$. Let $\Gamma_0 = \{\gamma \in \Gamma \mid \gamma\bar{\Omega} = \bar{\Omega}\}$. Then*

(1) *$D : \bar{\Omega} \to \mathbb{A}^2 - 0$ is a diffeomorphism onto its image.*

(2) *$\bar{\Omega}/\Gamma_0$ is a flat affine cylinder with geodesic boundary.*

*Proof.* Observe that $N = \tilde{N}$ is simply connected. Put $x_0 = D\tilde{x}_0$. It follows that $D : \tilde{N}\tilde{x}_0 \to Nx_0$ is a diffeomorphism. The complement of $N\tilde{x}_0$ in its closure $\overline{N\tilde{x}_0}$ consists of one-dimensional orbits for $N$, which are diffeomorphic to a ray in $\mathbb{A}^2 - 0$. Since $D$ is a local diffeomorphism on $\tilde{M}$, $\overline{N\tilde{x}_0}$ has precisely two such orbits in its closure, which map to their corresponding orbits in $\mathbb{A}^2 - 0$, see Figure 22. It follows that $D$ is injective on the closure $\overline{N\tilde{x}_0}$ and, in fact, $D : \overline{N\tilde{x}_0} \to \mathbb{A}^2 - 0$ is a diffeomorphism onto its image. This proves (1).

To prove (2), remark first that $\tilde{N}$ has at most finitely many orbits on the compact manifold $M$. This implies that there are only finitely many orbits of $\Gamma\tilde{N}$ on $\tilde{M}$. Since $\Gamma$ acts properly on $\tilde{M}$, it follows that every compact subset $\kappa$ of $\tilde{M}$ intersects only finitely many orbits of $\tilde{N}$. In particular, $\kappa$ intersects only finitely many components of $\Gamma\bar{\Omega}$. Therefore, $\Gamma\bar{\Omega}$ is closed in $\tilde{M}$. Hence, $\bar{\Omega}$ projects to a compact subset in $M$.

We may assume (by replacing $M$ with a finite covering manifold if necessary) that $h(\Gamma)$ is contained in $N$. Note then, if $\gamma \in \Gamma$ such that $\gamma\bar{\Omega} \cap \bar{\Omega} \neq \emptyset$ then $\gamma \in \Gamma_0$. In fact, since $h(\gamma) \in N$, there exists $\tilde{n} \in \tilde{N}$ such that $h(\gamma\tilde{n}) = 1$. Since $D$ is a diffeomorphism on $\bar{\Omega}$ and $\gamma\tilde{n}\bar{\Omega} \cap \bar{\Omega} \neq \emptyset$, we conclude that $\gamma\tilde{n}$ preserves both boundary components of $\bar{\Omega}$. Thus $\gamma\tilde{n}\Omega = \Omega$, and therefore $\gamma = n^{-1} \in \tilde{N}$. In particular, $\Gamma_0 = \Gamma \cap \tilde{N}$. Moreover, it follows that $\bar{\Omega}/\Gamma_0$ is the image of $\bar{\Omega}$ in $M = \tilde{M}/\Gamma$.

We thus proved that $\bar{\Omega}/\Gamma_0$ is compact. Since $\bar{\Omega}$ has two boundary components which are geodesic rays, $\bar{\Omega}/\Gamma_0$ must be a flat affine cylinder with geodesic circles as boundary components. This proves (2). $\qquad\square$

The development image of $\bar{\Omega}$ is a closed half space or a sector. This implies:

**Proposition 5.10.** *If the development image is $\mathbb{A}^2 - 0$ then the holonomy $h(\Gamma_0)$ is generated by an expansion.*

*Proof.* The proof of the previous proposition implies that $h(\Gamma_0)$ is contained in $N$. Since $\Gamma_0$ acts properly on $\overline{N\tilde{x}_0}$, and since $D$ is a diffeomorphism onto
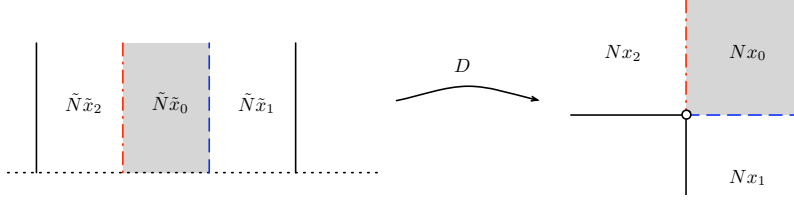
Figure 22. The orbits of $\tilde{N}$ in $\tilde{M}$.

its image $\overline{Nx_0}$, it follows that $h(\Gamma_0)$ acts properly and with compact quotient on the orbit closure $\overline{Nx_0}$. This implies that $h(\Gamma_0)$ has positive eigenvalues on one-dimensional orbits, and a fortiori, by properness on $\overline{Nx_0}$, it must be a group of expansions of $\mathrm{GL}(2,\mathbb{R})$ (see Figures 17 and 16).                      $\square$

**Final step in the proof.**   By the brick lemma (Proposition 5.9), $M$ decomposes as a finite union of copies of a flat affine cylinder $\bar{\Omega}/\Gamma_0$, which are glued along their boundary geodesics. Therefore, there exists a finite union $\bar{\mathcal{H}}$ of neighbouring copies of $\bar{\Omega}$ in $\tilde{M}$, such that the torus $M$ is obtained by identifying the two boundary geodesics of $\bar{\mathcal{H}}/\Gamma_0$ by an affine transformation $\tilde{B}$ in $\mathrm{Aff}(\tilde{M})$. Since $h(\tilde{B}) \in \mathrm{GL}^+(2,\mathbb{R})$ commutes with $N$, it follows that $B = h(\tilde{B})$ is contained in $N$ or $B \in \mathrm{R}_\pi N$. It follows that the development image of $\mathcal{H}$ must be $\mathbb{A}^2 - 0$ or $\bar{\mathcal{H}}_0$, the closed half space with the origin removed, respectively, see Figure 23. Hence, $\bar{\mathcal{H}}$ is affinely diffeomorphic to one of the strips $\bar{\mathcal{H}}_0^k$, which are defined in Section 4.4. Let $A \in \mathrm{GL}^+(2,\mathbb{R})$ be the expansion which generates $\Gamma_0$. Then $\bar{\mathcal{H}}/\Gamma_0$ is a flat affine cylinder $\mathcal{C}_A^k$, as defined in Example 4.18. Therefore, $M$ is affinely diffeomorphic to a flat affine torus $\mathcal{T}_{A,B,k}$ constructed in Example 4.15. In particular, $M$ is affinely diffeomorphic to a quotient of $\widetilde{\mathbb{A}^2 - 0}$, by a properly discontinuous subgroup

$$\Gamma = \langle \tilde{A}, \tilde{B}_k \rangle$$

of affine transformations in $\mathrm{Aff}(\widetilde{\mathbb{A}^2 - 0})$.
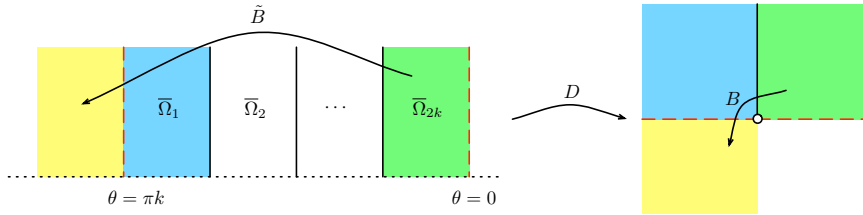


Figure 23. $\bar{\mathcal{H}} = \bar{\Omega}_1 \cup \cdots \cup \bar{\Omega}_{2k}$, $B = \mathrm{R}_\pi$.

**Corollary 5.11.** *Every non-homogenous flat affine two-torus $M$ is affinely diffeomorphic to a two-torus $\mathcal{T}_{A,B,k}$ (see Example 4.15), and $M$ is obtained by gluing a flat affine cylinder $\mathcal{C}_A^k$, where $A \in \mathrm{GL}^+(2,\mathbb{R})$ is an expansion, and $B \in \mathrm{GL}^+(2,\mathbb{R})$ has positive eigenvalues and commutes with $A$.*

# 6 The topology of the deformation space

In this section we describe the global and local structure of the deformation space $\mathfrak{D}(T^2, \mathbb{A}^2)$ of all flat affine structures on the two-torus. The deformation space decomposes into two *overlapping* subsets: the open subspace $\mathfrak{D}(T^2, \widetilde{\mathbb{A}^2 - 0})$ of structures modeled on the once punctured plane $\widetilde{\mathbb{A}^2 - 0}$ and the closed subspace $\mathfrak{D}_h(T^2, \mathbb{A}^2)$ of homogeneous flat affine structures. We describe the structure and topology of these two subspaces separately in Sections 6.3 to 6.4. In Section 6.5 we deduce our main result that the holonomy map for the deformation space $\mathfrak{D}(T^2, \mathbb{A}^2)$ is a local homeomorphism.

## 6.1  Flat affine connections

In this subsection we introduce flat affine connections. These provide another point of view on flat affine structures, which turns out to be particularly useful in the study of homogeneous flat affine manifolds.

An *affine connection* on the tangent bundle of $M$ is determined by a covariant differentiation operation on vector fields which is a $\mathbb{R}$-bilinear map

$$\nabla : \mathrm{Vect}(M) \times \mathrm{Vect}(M) \longrightarrow \mathrm{Vect}(M)$$
$$(X, Y) \longmapsto \nabla_X(Y)$$

and for $f \in C^\infty(M)$, satisfies

$$\nabla_{fX} Y = f \nabla_X Y, \qquad \text{and} \qquad \nabla_X(fY) = f \nabla_X Y + (Xf)Y$$

(where $Xf \in C^\infty(M)$ denotes the directional derivative of $f$ with respect to $X$). The connection is *torsion free* if and only if, for all $X, Y \in \mathrm{Vect}(M)$,

$$\nabla_X Y - \nabla_Y X = [X, Y] \,, \tag{6.1}$$

and it is *flat* if and only if the curvature tensor $R^\nabla$ vanishes. That is, if

$$R^\nabla(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = 0 \,. \tag{6.2}$$

**6.1.1  Correspondence with flat affine structures** Specifying an affine structure on $M$ is equivalent to giving a torsion free flat affine connection on

the tangent bundle of $M$. Indeed, let $M$ be a flat affine manifold. Then the affine structure defines a unique torsion free flat affine connection on $M$ by pulling back the canonical affine connection on $\mathbb{A}^n$ (that is, the usual derivative on $\mathbb{R}^n$) via a development map. Conversely, given any torsion free flat affine connection $\nabla$ on $M$, for each $p \in M$, the exponential map for $\nabla$ at $p$ is a connection preserving diffeomorphism from an open subset of the tangent vector space $T_pM$ (with the canonical flat affine connection) to a neighborhood of $p$, compare [47, VI. Theorem 7.2]. This gives rise to an atlas of locally affine coordinates and therefore determines a unique flat affine structure on $M$. Thus, there is a natural one to one correspondence

$$\mathfrak{S}(M, \mathbb{A}^n) \longleftrightarrow \{\text{torsion free flat affine connections on } M\} \qquad (6.3)$$

of the set of flat affine structures $\mathfrak{S}(M, \mathbb{A}^n)$ with a set of affine connections. An affine connection is called *complete* if all of its geodesics can be extended to infinity. Under the correspondence (6.3) complete affine structures are in bijection with complete affine connections.

Observe that the difference of two affine connections is a tensor field on $M$ and therefore the set of all affine connections forms an affine space.

**Example 6.1** (Flat connections form a closed subset of an affine space). Let $E$ denote the tangent bundle of the flat affine manifold $M$. Let $\nabla_0$ be the natural flat connection induced on $M$ by its flat affine structure. We choose $\nabla_0$ as a basepoint in the space of all affine connections on $M$. Every torsion free affine connection on $M$ is of the form $\nabla = \nabla_0 + S$, where $S \in \Gamma(S^2 E^* \otimes E)$ is a vector valued symmetric form on $M$. The set of all torsion free affine connections $\nabla$ on $M$ is thus an affine space modeled on the vector space $\Gamma(S^2 E^* \otimes E)$. Every torsion free *flat* affine connection on $M$ is of the form $\nabla = \nabla_0 + S$, where $S$ is contained in the closed subset $\mathcal{C}$ of $\Gamma(S^2 E^* \otimes E)$ defined by the equation (6.2), which encodes the vanishing of curvature.

The space of sections $\Gamma(S^2 E^* \otimes E)$ carries the $C^\infty$-topology of maps. This defines a topology on the space of torsion free affine connections.

**Proposition 6.2.** *The natural correspondence (6.3) of flat affine structures with flat torsion free affine connections is a homeomorphism. In particular, the space of flat affine structures $\mathfrak{S}(M, \mathbb{A}^n)$ is homeomorphic to the closed subset $\mathcal{C}$ in the tensor space $\Gamma(S^2 E^* \otimes E)$ as described above.*

*Proof.* Let us fix a flat affine structure on $M$ and let $\nabla_0$ be its compatible torsion free flat affine connection. Let $\nabla = \nabla_0 + S$ be another torsion free flat affine connection on $M$. In a local flat affine coordinate chart for $M$, $S \in \mathcal{C}$ is represented by a set of functions $\Gamma_{ij}^k$ which are called Christoffel symbols for $\nabla$,

see [47, III. Proposition 7.10]. We observe that the functions $\Gamma_{ij}^k$ also coincide with the coordinate representation of the tensor $S$. Therefore, a sequence $\nabla_n$ of affine connections is convergent if and only if the corresponding Christoffel symbols converge in all local flat affine coordinate systems.

Let $D_n$ be a sequence of development maps and consider the corresponding sequence of flat affine connections $\nabla_n$. Since the Christoffel symbols for $\nabla_n$ are polynomials in the first and second derivatives of $D_n$ (see [47]), convergence of $D_n$ implies convergence of $\nabla_n$. Therefore, the correspondence (6.3) is continuous.

Conversely, for any torsion free flat affine connection $\nabla$ on $M$, normal coordinate systems on $M$ define compatible coordinate charts for the flat affine structure defined by $\nabla$, see [47, VI. Theorem 7.2]. Normal coordinate systems for $\nabla$ are determined by an ordinary differential equation whose solutions depend smoothly on the Christoffel symbols for $\nabla$. Hence, the flat affine coordinate charts for $\nabla$ depend smoothly on $\nabla$. This shows that the correspondence (6.3) is a homeomorphism.                                                    □

### 6.1.2 Translation invariant flat affine connections

An affine connection $\nabla$ on the two-torus

$$T^2 = S^1 \times S^1$$

is called *translation invariant* if the group $S^1 \times S^1$ acts by affine transformations. Let $\nabla_0$ be the natural Riemannian flat affine connection on $T^2$. It is characterized by the property that the translation vector fields of the $S^1 \times S^1$-actions are parallel. Then any other connection $\nabla = \nabla_0 + S$ is translation invariant iff the covariant derivatives of the translation vector fields of the $S^1 \times S^1$-action are parallel with respect to $\nabla$. This condition is satisfied, if and only if the Christoffel symbols $S$ are constant functions in the flat coordinates for $\nabla_0$. Therefore, the set of all translation invariant torsion free flat affine connection is in bijection with the subset $\mathcal{C}(T^2, \mathbb{R})$ of $\mathcal{C}$ which consists of all *constant* (that is, of all $\nabla_0$-parallel) tensors contained in $\mathcal{C}$.

**Remark 6.3.** Equation (6.2) shows that $\mathcal{C}(T^2, \mathbb{R})$ is a *quadratic cone* in the vector space of symmetric bilinear maps $S^2 \mathbb{R}^2 \otimes \mathbb{R}^2$. Every element $S \in \mathcal{C}(\mathbb{R})$ represents a symmetric bilinear product

$$\cdot_\nabla : \ \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 , \ \ u \cdot_\nabla v := S(u, v) ,$$

which, for all $u, v, w$, satisfies the associativity relation

$$(u \cdot_\nabla v) \cdot_\nabla w = u \cdot_\nabla (v \cdot_\nabla w) .$$

This product defines a left-invariant flat affine connection on the abelian Lie group $\mathbb{R}^2$ by extending the covariant derivative from left-invariant vector fields to all vector fields. Indeed, there is a general correspondence of *associative,*

and more generally *left-symmetric* algebra products with left-invariant torsion free flat affine connections on Lie groups, see for example [5, Section 5.1]. Under this correspondence complete connections are represented by products which have the property that all maps $v \mapsto u \cdot_\nabla v$ have trace zero (compare [5, Corollary 5.7]).

We summarize this discussion by the following:

**Corollary 6.4.** (1) *The set of all translation invariant flat affine connections on $T^2$ is homeomorphic to a four-dimensional homogeneous quadratic cone $\mathcal{C}(T^2, \mathbb{R})$ in the six-dimensional vector space $S^2\mathbb{R}^2 \otimes \mathbb{R}^2$.*

(2) *The subset of* complete *translation invariant flat affine structures on $T^2$ is homeomorphic to a two-dimensional homogeneous quadratic cone in the vector space $\mathbb{R}^4$.*

In particular, one can deduce from (2) that the set of complete translation invariant flat affine connections on the two-torus is homeomorphic to $\mathbb{R}^2$. In view of Lemma 6.6, this gives yet another proof of the fact (cf. Example 3.26) that the deformation space of complete affine structures on the two-torus is homeomorphic to $\mathbb{R}^2$.

## 6.2 Translation invariant flat affine structures

The usual representation of the two-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ as a quotient of the vector group $\mathbb{R}^2$ by its integral lattice tacitly induces various extra structures. The translation action of the vector space $\mathbb{R}^2$ gives a simply transitive action of the abelian Lie group $S^1 \times S^1$ and the vector space structure on $\mathbb{R}^2$ descends to a compact abelian Lie group structure on $T^2$. Similarly, the ordinary flat affine structure on $\mathbb{R}^2$ induces the natural Riemannian flat affine structure on $T^2$ which is invariant by the translation group $S^1 \times S^1$.

**Definition 6.5.** A flat affine structure on $T^2 = S^1 \times S^1$ is called *translation invariant* if the group $S^1 \times S^1$ acts by affine transformations.

A translation invariant flat affine structure is thus compatible with the Lie group structure on $T^2$. In partciular, every flat affine torus with translation invariant flat affine structure is also a homogeneous flat affine torus. Note that the set of all translation invariant flat affine structures $\mathcal{T}(T^2, \mathbb{A}^2)$ corresponds to the set of translation invariant flat affine connections $\mathcal{C}(T^2, \mathbb{R})$ under the map (6.3).

**6.2.1 Relation with homogeneous flat affine tori** Let $M$ be a homogeneous flat affine two-torus, and $\mathrm{Aff}(M)_0$ the identity component of its affine

automorphism group. By the classification Theorem 5.1, the following two cases occur:

(1) Either the Lie group $\text{Aff}(M)_0$ is isomorphic to $S^1 \times S^1$ and it develops to an action of an affine Lie group as listed in Example 4.2.

(2) Or $M$ is affinely diffeomorphic to a Hopf torus. In this case, $\text{Aff}(M)_0$ contains a simply transitive group isomorphic to $S^1 \times S^1$, and this subgroup develops to the action of an affine Lie group of type A.

In particular, for every homogeneous flat affine two-torus $M$, the identity component $\text{Aff}(M)_0$ of the affine automorphism group of $M$ contains a two-dimensional compact abelian Lie group, which acts transitively and freely on $M$. This shows that every homogeneous flat affine two-torus is affinely diffeomorphic to a translation invariant flat affine torus.

Recall that the diffeomorphism group $\text{Diff}(T^2)$ acts on the set of all flat affine structures, and two flat affine structures on $T^2$ are called homotopic if they are equivalent by a diffeomorphism of $T^2$ which is homotopic to the identity.

**Lemma 6.6.** *Every homogeneous flat affine two-torus is homotopic (isotopic) to a unique translation invariant flat affine two-torus.*

*Proof.* Let $(f, M)$ be a marked homogeneous flat affine two-torus, where $f : T^2 \to M$ is a diffeomorphism. By the above remarks, we may choose a Lie subgroup $\mathcal{A}$ of $\text{Aff}(M)_0$ which acts simply transitively on $M$. The subgroup $\mathcal{A}$ is unique up to conjugacy in $\text{Aff}(M)_0$. We also choose a basepoint $m_0 \in M$. This fixes the structure of a compact abelian Lie group on $M$ which is isomorphic to $\mathcal{A}$. Then there exists a *unique* isomorphism of Lie groups $\phi : T^2 \to M$ such that $\phi^{-1} \circ f$ is homotopic to the identity of $T^2$. In other words, $(\phi, M)$ and $(f, M)$ are equivalent markings (cf. Section 3.4). By construction, the affine structure on $T^2$ induced by $\phi$ is translation invariant, and it is homotopic to the original homogeneous structure on $T^2$, which is induced by $(f, M)$. It is also independent of the choice of basepoint since $\mathcal{A}$ acts transitively on $M$ (compare also Example 3.34). Neither does it depend on the choice of the subgroup $\mathcal{A}$ in $\text{Aff}(M)_0$, since the conjugacy class of $\mathcal{A}$ in $\text{Aff}(M)_0$ is uniquely determined. In particular, this argument implies that every two translation invariant structures which are homotopic do coincide. This shows uniqueness. $\qquad\square$

The Lemma asserts that every orbit of the identity component $\text{Diff}_0(T^2)$ of the group of all diffeomorphisms $\text{Diff}(T^2)$ acting on homogeneous flat affine structures intersects the subset of translation invariant structures in precisely a single point. The proof also shows that on the subset of marked homogeneous tori which are in the complement of Hopf tori, we have a continuous projection

onto translation invariant tori. This proves that outside the Hopf tori the subset of translation invariant flat affine structures on $T^2$ defines a slice for the action of $\mathrm{Diff}_0(T^2)$ on the set of all homogeneous flat affine structures.

**6.2.2 Translation invariant development maps** We construct an explicit *continuous section* from the set of translation invariant flat affine structures to development maps. More specifically, we construct a continuous map

$$\mathcal{T}(T^2, \mathbb{A}^2) = \mathcal{C}(T^2, \mathbb{R}) \xrightarrow{\mathcal{E}} \mathrm{Dev}(T^2, \mathbb{A}^2) , \; S \mapsto D_S \qquad (6.4)$$

such that the development map $D_S$ defines an affine structure on $T^2$ which has associated flat affine connection $\nabla = \nabla_0 + S$. The construction is based on the relation of translation invariant flat affine structures with the set of commutative associative algebra products on $\mathbb{R}^2$ as follows:

**Example 6.7** (Associated étale affine representation). For $S \in \mathcal{C}(T^2, \mathbb{R})$, and $v \in \mathbb{R}^2$ we define an element

$$\bar{\rho}(v) = \left( \begin{array}{cc} S(v, \cdot) & v \\ 0 & 0 \end{array} \right) \in \mathfrak{aff}(2)$$

of the Lie algebra $\mathfrak{aff}(2)$ of the affine group $\mathrm{Aff}(2)$. In fact, the map $v \mapsto \bar{\rho}(v)$ is a Lie algebra homomorphism and the associated homomorphism of Lie groups

$$\rho = \rho_S : \mathbb{R}^2 \longrightarrow \mathrm{Aff}(2) , \; v \mapsto \rho(v) = \exp \bar{\rho}(v)$$

defines an affine representation of the Lie group $\mathbb{R}^2$ on $\mathbb{A}^2$ which is étale in $0 \in \mathbb{A}^2$ (cf. Definition 3.9 and also the discussion in [6, Section 2.1]).

Let $\nabla$ be the translation invariant flat affine connection on $\mathbb{R}^2$ which is represented by $S \in \mathcal{C}(T^2, \mathbb{R})$. The orbit map of the étale representation $\rho_S$ is

$$D_S = o_S : \mathbb{R}^2 \longrightarrow \mathbb{A}^2 , \; v \mapsto \rho_S \cdot 0$$

and it is a development map for a translation invariant flat affine structure on $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$ with associated affine connection $\nabla$. (In fact, $D_S$ is also a *frame preserving* development map. Compare Example 3.24 and Section 3.4.1.) Since, $D_S$ depends smoothly on $S$,

$$\mathcal{E}(S) = D_S$$

defines the required continuous section.

Note that the holonomy homomorphism $h = h^\nabla : \mathbb{Z}^2 \to \mathrm{Aff}(2)$ for $D_S$ satisfies

$$h^\nabla(\gamma) = \rho(\gamma), \text{ for all } \gamma \in \mathbb{Z}^2 .$$

We state without proof:

**Lemma 6.8.** *The continuous map* $\mathcal{C}(T^2, \mathbb{R}) \to \mathrm{Hom}(\mathbb{Z}^2, \mathrm{Aff}(2))$, $\nabla \mapsto h^\nabla$, *is locally injective.*

## 6.3 The space of structures modeled on $(\mathbb{A}^2 - 0, \mathrm{GL}(2, \mathbb{R}))$

Here we discuss in detail the subspace of the deformation space of flat affine structures on the two-torus which consists of structures which have the once-punctured plane as development image. Our main observation is that the holonomy map is a local homeomorphism on such structures.

**6.3.1 The holonomy map** We consider first the subgeometry of structures which are modeled on the universal covering of the once-punctured plane. The topology of the deformation space of such structures is completely controlled by the holonomy map into the space of conjugacy classes of homomorphisms (the "character variety"):

**Theorem 6.9.** *The holonomy map*

$$\mathfrak{D}(T^2, \widetilde{\mathbb{A}^2 - 0}) \xrightarrow{hol} \mathrm{Hom}(\mathbb{Z}^2, \widetilde{\mathrm{GL}}^+(2, \mathbb{R}))/\widetilde{\mathrm{GL}}(2, \mathbb{R})$$

*embeds the deformation space homeomorphically as an open connected subset of the character variety.*

*Proof.* Note, since $T^2$ is orientable, the holonomy takes values in $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$. The map *hol* is injective, by Theorem 5.6. Since *hol* is also continuous and open, it is a homeomorphism onto an open subset. Connectedness of the deformation space will follow from the considerations in Section 6.3.2 below. ∎

Now we look at the deformation space of structures which modeled on the once-punctured plane. For such structures the holonomy map is not injective, as we already remarked in Example 5.5. However, as we show now at least *locally* the topology of the deformation space of $(\mathbb{A}^2 - 0, \mathrm{GL}(2, \mathbb{R}))$ structures is fully controlled by the character variety:

**Corollary 6.10.** *The holonomy map*

$$\mathfrak{D}(T^2, \mathbb{A}^2 - 0) \xrightarrow{hol} \mathrm{Hom}(\mathbb{Z}^2, \mathrm{GL}^+(2, \mathbb{R}))/\mathrm{GL}(2, \mathbb{R})$$

*is a local homeomorphism onto its image, which is a connected open subset in the character variety.*

*Proof.* Since the subgeometry

$$(\widetilde{\mathbb{A}^2 - 0}, \widetilde{\mathrm{GL}}(2, \mathbb{R})) \longrightarrow (\mathbb{A}^2 - 0, \mathrm{GL}(2, \mathbb{R}))$$

is a covering, the induced map on deformation spaces (cf. Section 3.1.2)

$$\mathfrak{D}(T^2, \widetilde{\mathbb{A}^2 - 0}) \longrightarrow \mathfrak{D}(T^2, \mathbb{A}^2 - 0)$$

is a homeomorphism by Lemma 3.22. The commutative diagram (3.8) for the subgeometry takes the form

$$
\begin{array}{ccc}
\mathfrak{D}(T^2, \widetilde{\mathbb{A}^2 - 0}) & \xhookrightarrow{\ hol\ } & \mathrm{Hom}(\mathbb{Z}^2, \widetilde{\mathrm{GL}}^+(2, \mathbb{R}))/\widetilde{\mathrm{GL}}(2, \mathbb{R}) \\
{\scriptstyle \approx}\Big\downarrow & & \Big\downarrow \\
\mathfrak{D}(T^2, \mathbb{A}^2 - 0) & \xrightarrow{\ hol\ } & \mathrm{Hom}(\mathbb{Z}^2, \mathrm{GL}^+(2, \mathbb{R}))/\mathrm{GL}(2, \mathbb{R}) \,.
\end{array}
\qquad (6.5)
$$

Note that, by Theorem 6.9, the top horizontal map is a topological embedding. Furthermore, by Corollary A.8, the right vertical map is a local homeomorphism. We deduce that the bottom map *hol* for $\mathfrak{D}(T^2, \mathbb{A}^2 - 0)$ is locally injective, and therefore it is a local homeomorphism onto an open subset. $\square$

In the situation of Corollary 6.10, all local topological properties of the deformation space are reflected in the character variety and also vice versa. For instance, singularities in the character variety give rise to singularities in the deformation space, as is the case in Example 3.29. *This shows that* $\mathfrak{D}(T^2, \mathbb{A}^2 - 0)$ *is not Hausdorff, and it is not even a* $\mathrm{T}_1$*-topological space.*

**6.3.2 Cartography of the deformation space** Important strata in the deformation space

$$\mathfrak{D}(T^2, \widetilde{\mathbb{A}^2 - 0})$$

arise from the orbit types of the action of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ on the image of the holonomy map

$$\mathsf{hol} : \mathrm{Dev}(T^2, \widetilde{\mathbb{A}^2 - 0}) \longrightarrow \mathrm{Hom}(\mathbb{Z}^2, \widetilde{\mathrm{GL}}^+(2, \mathbb{R})) \,.$$

We introduce several such strata and describe their topological relations with each other. We use this information to establish the connectedness of the deformation space.

**Overview** According to the classification theorem, tori which are modeled on $\widetilde{\mathbb{A}^2 - 0}$ fall into three main classes distinguished by their development images. Namely the classes are formed by structures which have development image equivalent to either

(1) the once punctured plane $\widetilde{\mathbb{A}^2 - 0}$ ("complete structures"),

(2) or a sector $\mathcal{Q}$,

(3) or the open half space $\mathcal{H}$.

The structures with development image $\widetilde{\mathbb{A}^2 - 0}$ comprise non-homogeneous flat affine tori and the homogeneous structures which arise from (lifts of) étale representations of type $\mathsf{A}$. The latter two strata arise from (the lifts of) étale representations of type $\mathsf{B}$ and $\mathsf{C}_1$ respectively (see Example 4.2 for notation). Therefore all corresponding tori in these two strata are homogeneous.

Another decomposition of the deformation space is obtained by considering the subset $\mathfrak{T}$ of non-homogeneous structures and its complementary subspace $\mathfrak{D}_h(T^2, \widetilde{\mathbb{A}^2 - 0})$ consisting of homogeneous structures. The space of non-homogeneous structures can be decomposed into connected components parametrized by the level of a non-homogeneous structure. The subset of homogeneous structures is connected. The subspace $\mathfrak{D}_h(T^2, \widetilde{\mathbb{A}^2 - 0})$ contains a two-dimensional stratum $\mathfrak{H}$ of Hopf tori as a distinguished subset. Non-homogenous structures are connected to homogeneous ones only along the space $\mathfrak{H}$ of Hopf tori.

**Hopf tori**   Recall that a torus which is modeled on $\widetilde{\mathbb{A}^2 - 0}$ is called a *Hopf torus* if its holonomy is contained in the center of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$. The holonomy homomorphisms of marked Hopf tori form the closed subset of fixed points for the $\widetilde{\mathrm{GL}}(2, \mathbb{R})$-conjugation action on the holonomy image of $\widetilde{\mathbb{A}^2 - 0}$ - structures. Therefore, the Hopf tori form a closed subset

$$\mathfrak{H} \subset \mathfrak{D}(T^2, \widetilde{\mathbb{A}^2 - 0}) \, .$$

All Hopf tori are derived from the étale affine representation of type $\mathsf{A}$ as follows. Let

$$o : \mathbb{R}^2 \longrightarrow \mathbb{A}^2 - 0 \, , \quad (t, \theta) \mapsto \exp(t)(\cos\theta, \sin\theta)$$

be the orbit map associated to the representation $\mathsf{A}$. For $k_1, k_2 \in \mathbb{Z}$ and $\lambda_1, \lambda_2 > 0$, let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map, which satisfies

$$\phi(e_1) = (\log\lambda_1, k_1\pi) \, , \ \ \phi(e_2) = (\log\lambda_2, k_2\pi) \, .$$

Then development maps of the form

$$D = o \circ \phi : \mathbb{R}^2 \to \mathbb{A}^2 - 0$$

define a two-parameter family of marked Hopf tori

$$\mathcal{H}_{\lambda_1, \lambda_2, k_1, k_2} \quad , \ \ (\log\lambda_1)k_2 - (\log\lambda_2)k_1 \neq 0.$$

(See Section 3.3.5 for the general construction.) The corresponding holonomy homomorphisms $h : \mathbb{Z}^2 \to \mathrm{Hom}(\mathbb{Z}^2, \widetilde{\mathrm{GL}}(2, \mathbb{R}))$ satisfy

$$h(e_i) = \, \mathrm{diag}(\lambda_i)\,\tau^{k_i} \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R}) \, .$$

(Here $e_1, e_2$ denote generators of $\mathbb{Z}^2$, and $\mathrm{diag}(\lambda) \in A$ the diagonal matrix which has both diagonal entries equal to $\lambda$.) Observe that every marked Hopf torus is equivalent in the deformation space to precisely one of these tori. Forgetting about the marking, we note that every Hopf torus is affinely equivalent to a torus of the form $\mathcal{H}_{\lambda_1, \lambda_2, k, 0}$, where we call $k = \gcd(k_1, k_2) \neq 0$ the *level* of $\mathcal{H}$.

For fixed $k_1, k_2 \in \mathbb{Z}$, the set of all $\mathcal{H}_{\lambda_1, \lambda_2, k_1, k_2}$ parametrizes a closed (and also connected subset) of Hopf tori $\mathfrak{H}_{k_1, k_2}$, and the subset of all Hopf tori decomposes as

$$\mathfrak{H} \;=\; \bigcup_{(k_1, k_2) \neq 0} \mathfrak{H}_{k_1, k_2} \;\subset\; \mathfrak{D}(T^2, \widetilde{\mathbb{A}^2 - 0}) \,.$$

**Non-homogeneous tori**   We let

$$\mathfrak{T} \subset \; \mathfrak{D}(T^2, \widetilde{\mathbb{A}^2 - 0})$$

denote the subset of non-homogenous structures. Every marked manifold $\mathcal{T}$ which represents an element of $\mathfrak{T}$ is equivalent as an $(\widetilde{\mathbb{A}^2 - 0}, \widetilde{\mathrm{GL}}^+(2, \mathbb{R}))$-manifold to a torus

$$\mathcal{T}_{A,B,k}$$

as constructed in Corollary 5.11. Here $A \in \mathrm{GL}^+(2, \mathbb{R})$ is an expansion and $B \in \mathrm{GL}^+(2, \mathbb{R})$ is upper triangular and commuting with $A$. We call the number $k \in \mathbb{Z} - \{0\}$ the *level* of $\mathcal{T}$, respectively the level of its class in $\mathfrak{T}$. Since the level cannot be zero, claim (2) of Lemma A.4 implies that *the set $\mathfrak{T}$ is indeed an open subset of the deformation space.*

Let $\mathfrak{T}_k$ denote the set of all elements in $\mathfrak{T}$ of level $k$. We have the disjoint decomposition

$$\mathfrak{T} = \bigcup_{k \in \mathbb{Z} - \{0\}} \mathfrak{T}_k \;.$$

Proposition A.5 implies that all subsets $\mathfrak{T}_k$ and their complements are closed subsets of $\mathfrak{T}$.

**The closure of non-homogeneous tori**   We show now that the boundary of the set of non-homogeneous structures $\mathfrak{T}$ in the deformation space is formed by Hopf tori.

**Proposition 6.11.** *The closure of $\mathfrak{T}_k$ in $\mathfrak{D}(T^2, \widetilde{\mathbb{A}^2 - 0})$ is $\mathfrak{T}_k \cup \mathfrak{H}_k$.*

*Proof.* Note first that the Hopf tori $\mathcal{H}_{\lambda_1,\lambda_2,k,0} \in \mathfrak{H}_k$, are in the closure of the elements $\mathcal{T}_{A,B,k}$ of $\mathfrak{T}_k$. (Just deform $A$ and $B$ to dilations.) It remains to show that every homogeneous torus $M_o$ in the closure of $\mathfrak{T}_k$ is a Hopf torus: Let $M_\epsilon$ equivalent to $\mathcal{T}_{A,B,k}$ be a marked non-homogeneous torus of level $k \neq 0$ which is in the vicinity of $M_o$ in the deformation space. Let $h_\epsilon$ denote its holonomy homomorphism. We assume that $h_\epsilon$ converges to $h_o$ in the space of conjugacy classes of homomorphisms. The holonomy group of $M_\epsilon$ is generated by $h_\epsilon(e_i) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, $i = 1, 2$. The conjugacy class $\mathcal{C}h_\epsilon(e_i)$ is thus in the vicinity of the class $\mathcal{C}h_o(e_i)$. Proposition A.5 implies that the projections $\mathsf{p}(h_o(e_i)) \in \mathrm{GL}^+(2,\mathbb{R})$ are conjugate to an element of $AN$. Since $M_o$ is homogeneous, $M_o$ is either a Hopf torus or $\mathrm{lev}\, h_o(e_1) = \mathrm{lev}\, h_o(e_2) = 0$ (in which case $M_o$ has development image $\mathcal{H}$ or $\mathcal{Q}$). In the latter case, if $M_\epsilon$ is close enough, we must have $\mathrm{lev}\, h_\epsilon(e_1) = \mathrm{lev}\, h_\epsilon(e_2) = 0$, again by Proposition A.5. This contradicts the fact that the level of the non-homogeneous torus $M_\epsilon$ is different from zero. Therefore, $M_o$ is a Hopf torus.                           $\square$

**Homogeneous tori modeled on** $\mathbb{A}^2 - 0$    The subset of homogeneous structures

$$\mathfrak{D}_h(T^2, \widetilde{\mathbb{A}^2 - 0}) \subset \mathfrak{D}(T^2, \widetilde{\mathbb{A}^2 - 0})$$

decomposes into three strata $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}_1$, which are distinguished according to the development images $\widetilde{\mathbb{A}^2 - 0}$, $\mathcal{Q}$ and $\mathcal{H}$ respectively. These structures arise from the étale affine representations of the abelian Lie group $\mathbb{R}^2$ of type $\mathsf{A}$, $\mathsf{B}$ and $\mathsf{C}_1$ respectively. Moreover, it follows (using the construction in Section 3.3.5) that the three strata are continuous images of homogeneous spaces via maps

$$\mathrm{GL}(2,\mathbb{R})/N \longrightarrow \mathfrak{D}(T^2, \widetilde{\mathbb{A}^2 - 0}),$$

where $N$ describes the group of those automorphisms of $\mathbb{R}^2$ which are induced by the conjugation action of the normalizers in $\mathrm{GL}(2,\mathbb{R})$ for the groups $\mathsf{A}$, $\mathsf{B}$ and $\mathsf{C}_1$. (The normalizers are listed in Lemma 4.3.) In particular, it follows that the strata $\mathfrak{A}$ and $\mathfrak{B}$ are images of connected four-dimensional manifolds, while $\mathfrak{C}_1$ is a connected manifold of dimension three.

Note that structures in $\mathfrak{A}$ may be continuously deformed to structures in $\mathfrak{C}_1$, as follows from (4) of Lemma A.4. Compare also Figure 9. Similarly structures in $\mathfrak{B}$ can be deformed to structures in $\mathfrak{C}_1$, see Figure 4. This shows in particular that $\mathfrak{D}_h(T^2, \widetilde{\mathbb{A}^2 - 0})$ is connected.

**Connectedness**    The deformation space

$$\mathfrak{D}(T^2, \widetilde{\mathbb{A}^2 - 0}) = \mathfrak{T} \cup \mathfrak{D}_h(T^2, \widetilde{\mathbb{A}^2 - 0})$$

is connected. Indeed, by Proposition 6.11 every marked non-homogeneous flat affine torus in $\mathfrak{T}$ can be deformed to a Hopf torus contained in some $\mathfrak{H}_{k_1,k_2}$. Since $\mathfrak{H}_{k_1,k_2}$ is a subset of the connected space $\mathfrak{D}(T^2, \widetilde{\mathbb{A}^2 - 0})_h$ it follows that $\mathfrak{D}(T^2, \widetilde{\mathbb{A}^2 - 0})$ is connected.

## 6.4 The subspace of homogeneous structures

We describe now the properties of the subset $\mathfrak{D}_h(T^2, \mathbb{A}^2)$ of homogeneous flat affine structures as a subspace of the deformation space of all flat affine structures on $T^2$. Since the non-homogeneous structures form an open subset the space $\mathfrak{D}_h(T^2, \mathbb{A}^2)$ is closed. The complement of Hopf tori

$$\mathfrak{D}_h(T^2, \mathbb{A}^2) - \mathfrak{H}$$

forms a dense subset which is also open in the deformation space of all structures $\mathfrak{D}(T^2, \mathbb{A}^2)$. We established in the previous subsections:

**Proposition 6.12.** *The set of all homogeneous structures $\mathfrak{D}_h(T^2, \mathbb{A}^2)$ is the continuous and injective image of the quadratic cone $\mathcal{C}(T^2, \mathbb{R})$ under the map $\mathcal{E}$ in* (6.4). *The map is a homeomorphism in the complement of Hopf tori.*

In particular:

**Corollary 6.13.** *The deformation space $\mathfrak{D}_h(T^2, \mathbb{A}^2)$ of all homogeneous flat affine structures on the two-torus contains the complement of Hopf tori*

$$\mathfrak{D}_h(T^2, \mathbb{A}^2) - \mathfrak{H}$$

*as a dense open subset, which is a Hausdorff space and homeomorphic to a Zariski-open subset in a four-dimensional quadratic cone in $\mathbb{R}^6$.*

Note that the space of complete affine structures $\mathfrak{D}_c(T^2, \mathbb{A}^2)$ forms a two-dimensional closed subcone which is homeomorphic to $\mathbb{R}^2$, see [6].

**6.4.1 The action of the linear group on translation invariant structures and conjugacy of étale affine groups** The linear group $\mathrm{GL}(2, \mathbb{R})$ naturally acts on the variety $\mathcal{C}(T^2, \mathbb{R})$ of commutative and associative algebra products on $\mathbb{R}^2$. The orbits of this action correspond to the isomorphism classes of algebra products. Since the section map

$$\mathcal{E} : \mathcal{C}(T^2, \mathbb{R}) \longrightarrow \mathfrak{D}_h(T^2, \mathbb{A}^2)$$

is a continuous bijection this constructs a *natural* induced action of $\mathrm{GL}(2, \mathbb{R})$ on the deformation space of homogeneous structures $\mathfrak{D}_h(T^2, \mathbb{A}^2)$ which is continuous on the complement of Hopf tori. This group action may be used to

reveal some of the topology of $\mathfrak{D}_h(T^2, \mathbb{A}^2)$ and the possible deformations of structures.

Recall the classification of étale affine representations which is described in Section 4.1. Each orbit of $\mathrm{GL}(2,\mathbb{R})$ in $\mathfrak{D}_h(T^2, \mathbb{A}^2)$ corresponds to exactly one of the affine conjugacy classes of abelian almost simply transitive groups of affine transformations on $\mathbb{A}^2$. We label the orbits accordingly with the symbols $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{D}$ and $\mathsf{T}$. The decomposition of $\mathfrak{D}_h(T^2)$ into the six orbit types of $\mathrm{GL}(2,\mathbb{R})$ defines a natural stratification on $\mathfrak{D}_h(T^2)$ into manifolds which are homogeneous spaces of $\mathrm{GL}(2,\mathbb{R})$, and each orbit is a subcone of $\mathfrak{D}_h(T^2)$. Each such stratum may be also computed as the induced image of the subgeometry which is defined by the corresponding étale affine representation, see the examples in Section 6.3.2, as well as Section 3.3.5 and Lemma 4.3.

The closure of each stratum consists of strata of lower dimensions and contains the unique closed stratum $\mathsf{T}$, which is a point. There are two open strata of dimension four labeled $\mathfrak{A}$ and $\mathfrak{B}$, which correspond to homogeneous flat affine structures whose development images are the punctured plane, and the sector respectively. In their closure are the three-dimensional orbits $\mathfrak{C}_1$ and $\mathfrak{C}_2$, whose corresponding flat affine structures develop into the halfplane. The complete structures correspond to a two dimensional orbit $\mathfrak{D}$ and the translation structure $\mathsf{T}$. We say that the orbit $\mathcal{O}_1$ degenerates to the orbit $\mathcal{O}_2$ if $\mathcal{O}_2$ is in the closure of $\mathcal{O}_1$. Degeneration induces a partial ordering on the strata of $\mathfrak{D}_h(T^2, \mathbb{A}^2)$, with the translation action the unique minimal point. By the theorem of Hilbert-Mumford if $\mathcal{O}_2$ degenerates to $\mathcal{O}_1$ then there exists a one-parameter group $\lambda : \mathbb{R} \to \mathrm{GL}(2,\mathbb{R})$ such that $\lim_{t\to 0} \lambda(t)o_2 \in \mathcal{O}_2$. Therefore, every degeneration may be constructed explicitly as a limit of a curve of flat affine structures in the stratum. Moreover, every point of $\mathfrak{D}_h(T^2, \mathbb{A}^2)$ directly degenerates to the translation structure, compare, for example, Figure 7.

The graph shown in Figure 24 describes all possible degenerations in the orbit stratification of $\mathfrak{D}_h(T^2, \mathbb{A}^2)$ with respect to the natural action of $\mathrm{GL}(2,\mathbb{R})$.
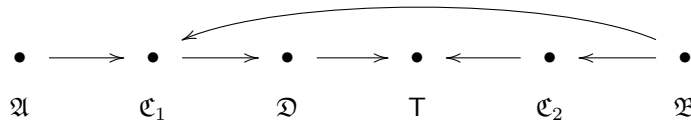


Figure 24. Degenerations of $\mathrm{GL}(2,\mathbb{R})$-orbit types in the deformation space.

These degenerations are illustrated in Figures 1-5, Figure 7 and Figure 9.

## 6.5 The deformation space of all flat affine structures on the two-torus

Our main result is the following.

**Theorem 6.14.** *The holonomy map for flat affine structures on the two-torus*

$$hol : \mathfrak{D}(T^2, \mathbb{A}^2) \to \mathrm{Hom}(\mathbb{Z}^2, \mathrm{Aff}(2))/\mathrm{Aff}(2)$$

*is a local homeomorphism onto an open connected subset of the character variety.*

**The holonomy map for flat affine structures** For the proof of Theorem 6.14 we consider first the subgeometry of $(\mathbb{A}^2 - 0, \mathrm{GL}(2, \mathbb{R}))$-structures and its induced map on deformation spaces (cf. Section 3.1.2):

**Proposition 6.15.** *The induced map on deformation spaces*

$$\mathfrak{D}(T^2, \mathbb{A}^2 - 0) \longrightarrow \mathfrak{D}(T^2, \mathbb{A}^2)$$

*is an embedding onto an open subset $\mathfrak{U}_o$ of the space $\mathfrak{D}(T^2, \mathbb{A}^2)$. Moreover, the holonomy map for $\mathfrak{D}(T^2, \mathbb{A}^2)$ restricts to a local homeomorphism on this subset.*

*Proof.* The commutative diagram (3.8) for the subgeometry takes the form

$$
\begin{array}{ccc}
\mathfrak{D}(T^2, \mathbb{A}^2 - 0) & \xrightarrow{\ hol\ } & \mathrm{Hom}(\mathbb{Z}^2, \mathrm{GL}^+(2, \mathbb{R}))/\mathrm{GL}(2, \mathbb{R}) \\
\downarrow & & \downarrow \\
\mathfrak{D}(T^2, \mathbb{A}^2) & \xrightarrow{\ hol\ } & \mathrm{Hom}(\mathbb{Z}^2, \mathrm{Aff}(2))/\mathrm{Aff}(2)\,.
\end{array}
$$

The image of $\mathfrak{D}(T^2, \mathbb{A}^2 - 0)$ in $\mathfrak{D}(T^2, \mathbb{A}^2)$ consists of precisely those structures in $\mathfrak{D}(T^2, \mathbb{A}^2)$ whose linear part of the holonomy contains an expansion. Therefore, the image $\mathfrak{U}_o$ of the induced map is open in $\mathfrak{D}(T^2, \mathbb{A}^2)$. The left vertical map is clearly injective. Note further that the operation of taking the linear part of a homomorphism defines a continuous section of the right vertical map which is defined on the holonomy image $hol(\mathfrak{U}_o)$. The latter set is open in $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{Aff}(2))/\mathrm{Aff}(2)$. By Corollary 6.10, the upper map $hol$ is a local homeomorphism. Therefore, the right vertical map is a topological embedding, and the lower map $hol$ is a local homeomorphism on $\mathfrak{U}_o$. $\qquad\square$

Below we construct an open neighborhood $\mathfrak{U}_1$ of the translation structure $\mathsf{T} \in \mathfrak{D}(T^2, \mathbb{A}^2)$, which has the following properties:

(1) $\mathfrak{U}_1 \subset \mathfrak{D}_h(T^2, \mathbb{A}^2)$ is contained in the subset of homogeneous flat affine structures,

(2) the restriction of the holonomy map $hol : \mathfrak{U}_1 \to \mathrm{Hom}(\mathbb{Z}^2, \mathrm{Aff}(2))/\mathrm{Aff}(2)$ is injective,

(3) $\mathfrak{D}(T^2, \mathbb{A}^2) = \mathfrak{U}_o \cup \mathfrak{U}_1$.

Together with Proposition 6.15 this shows that

$$hol : \mathfrak{D}(T^2, \mathbb{A}^2) \to \mathrm{Hom}(\mathbb{Z}^2, \mathrm{Aff}(2))/\mathrm{Aff}(2)$$

is locally injective and therefore finishes the proof of Theorem 6.14.

We observe the following refinement of Proposition 6.11:

**Proposition 6.16.** *The closure of the subset $\mathfrak{T}$ of non-homogeneous structures in the deformation space $\mathfrak{D}(T^2, \mathbb{A}^2)$ consists of Hopf tori.*

*Proof.* Suppose there is a sequence of non-homogeneous marked tori $M_i$ which converge in the deformation space $\mathfrak{D}(T^2, \mathbb{A}^2)$ to a flat affine torus $M$. Let $h_i : \mathbb{Z}^2 \to \mathrm{Aff}(2)$ be their corresponding holonomy homomorphisms. By Corollary 5.11, we may assume that the linear parts of the $h_i$ are contained in the group of upper triangular matrices $AN \cup -AN$, where $AN$ is the index two subgroup with positive diagonal entries. Now if $M$ is homogeneous and has development image different from the once-punctured plane, the linear parts of all $h_i$ are contained in $AN$ for sufficiently large $i$. Let $D_i$ be a corresponding sequence of development maps for the $M_i$ which converges to a development map $D$ which represents $M$. Since $M$ is not modeled on the once-punctured plane the development map $D$ is injective. By Example 3.14, there is a neighborhood of $D$ in the space of development maps such that $D$ is injective on the fundamental domain for the standard action of $\mathbb{Z}^2$ on $\mathbb{R}^2$. However, the development maps $D_i$ are not injective on this fundamental domain by construction of the non-homogeneous tori $M_i$, see Example 4.15. This contradicts the fact that the $D_i$ converge to $D$. The claim now follows from Proposition 6.11. $\qquad\square$

Now the construction of $\mathfrak{U}_1$ goes as follows: Following the notation in Appendix A, define $U_\epsilon = \{g \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R}) \mid |\theta(g)| < \epsilon\}$. By the proof of Proposition A.7, we may choose an open set

$$\mathcal{U}_\epsilon \subset \mathrm{Hom}(\mathbb{Z}^2, \widetilde{\mathrm{GL}}^+(2, \mathbb{R}))/\widetilde{\mathrm{GL}}(2, \mathbb{R}) \,,$$

where $\mathcal{U}_\epsilon$ is of the form $\mathcal{C}(U_\epsilon \times U_\epsilon)$ such that the projection

$$\mathcal{U}_\epsilon \longrightarrow \mathrm{Hom}(\mathbb{Z}^2, \mathrm{GL}^+(2, \mathbb{R}))/\mathrm{GL}(2, \mathbb{R})$$

is injective. The holonomy preimage $hol^{-1}(\mathcal{U}_\epsilon)$ is a non-empty open subset of $\mathfrak{D}(T^2, \widetilde{\mathbb{A}^2 - 0})$ which contains certain homogeneous structures of type $\mathfrak{A}$,

and the strata $\mathfrak{B}$ and $\mathfrak{C}_1$. It corresponds to a non-empty open subset $\mathcal{V}_\epsilon$ of $\mathfrak{D}(T^2, \mathbb{A}^2 - 0)$ such that the restriction

$$hol : \mathcal{V}_\epsilon \longrightarrow \operatorname{Hom}(\mathbb{Z}^2, \operatorname{GL}^+(2, \mathbb{R}))/\operatorname{GL}(2, \mathbb{R})$$

is injective. Let $\mathcal{M} = \mathfrak{D}(T^2, \mathbb{A}^2 - 0) - \mathcal{V}_\epsilon$ be the complement (containing the non-homogeneous flat affine tori and also homogeneous structures of type $\mathfrak{A}$). Then we observe that $\mathcal{M}$ is closed not only in $\mathfrak{D}(T^2, \mathbb{A}^2 - 0)$, but also in $\mathfrak{D}(T^2, \mathbb{A}^2)$. (Indeed, this follows since the closure of the space $\mathfrak{T}$ of non-homogeneous tori is contained in the space $\mathfrak{H}$ of Hopf tori.) Now we put $\mathfrak{U}_1 = \mathfrak{D}(T^2, \mathbb{A}^2) - \mathcal{M}$.

# A  Conjugacy classes in the universal covering group of $\mathrm{GL}(2, \mathbb{R})$

Let $\mathrm{GL}^+(2, \mathbb{R})$ be the group of $2 \times 2$ matrices with positive determinant, and let

$$\mathsf{p} : \widetilde{\mathrm{GL}}^+(2, \mathbb{R}) \to \mathrm{GL}^+(2, \mathbb{R})$$

be its universal covering group.

**Iwasawa decomposition**  Recall the Iwasawa decomposition

$$\mathrm{GL}^+(2, \mathbb{R}) = KAN \ ,$$

where $K = \mathrm{SO}(2, \mathbb{R})$ is the subgroup of rotations, $A$ is the group of diagonal matrices with positive entries, and $N$ the group of unipotent upper triangular matrices. Furthermore, we let $D$ be the central subgroup of $\mathrm{GL}^+(2, \mathbb{R})$ contained in $A$ which consists of all elements of $A$ with identical diagonal entries.

Let $\tilde{K} \to K$ be the universal covering of the rotation group. There is an induced Iwasawa decomposition

$$\widetilde{\mathrm{GL}}^+(2, \mathbb{R}) = \tilde{K}AN \ ,$$

where $A$ and $N$ are considered as subgroups of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$.

Let $\mathcal{Z}$ be the subgroup of $\tilde{K}$, which is mapped by the covering projection onto $\{+1, -1\} = \{\mathrm{E}_2, \mathrm{R}_\pi\} \subset \mathrm{SO}(2, \mathbb{R})$. Note that the center of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ consists of the subgroup $D$ extended by the group $\mathcal{Z}$. We choose a generator $\tau \in \tilde{K}$ for the infinite cyclic group $\mathcal{Z}$. Then $\mathsf{p}(\tau) = \mathrm{R}_\pi \, (= -\mathrm{E}_2)$ and the element $\tau^2 \in \mathcal{Z}$ generates the kernel of the covering projection $\mathsf{p}$.

**The rotation angle function**  We consider the diffeomorphism $\theta : \tilde{K} \to \mathbb{R}$ which satisfies $\theta(1) = 0$ and the relation

$$\mathsf{p} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \ .$$

Using the Iwasawa decomposition we construct an angular map

$$\theta : \widetilde{\mathrm{GL}}^+(2, \mathbb{R}) \to \mathbb{R}$$

by extending $\theta : \tilde{K} \to \mathbb{R}$. This means that $\theta(g) = \theta(k)$, where $g \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ has decomposition $g = kan$. Geometrically, $\theta(g)$ thus is the angle of rotation or polar angle for the image $\mathsf{p}(g)(e_1)$ of the first standard basis vector $e_1$. More specifically, when considering the action of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ on $\widetilde{\mathbb{A}^2 - 0}$ (see

Section 4.3.2), the function $\theta$ can also be read off as the $\theta$-coordinate of

$$g \cdot (r, 0) = (s, \theta(g)) \in \widetilde{\mathbb{A}^2 - 0} .$$

The following properties of the function $\theta$ are easy to verify:

**Lemma A.1.** *Let* $g, h \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$. *Then*

(1) $\theta(g) = 0$ *if and only if* $g \in AN$.

(2) $\theta(kg) = \theta(k) + \theta(g)$, *for all* $k \in \tilde{K}$; $\theta(\tau^m) = m\pi$.

(3) $|\theta(gh) - \theta(g) - \theta(h)| < \pi$.

(4) $|\theta(g) - \theta(g^{-1})| < \pi$.

(5) $|\theta(gkg^{-1}) - \theta(k)| < \pi$, *for all* $k \in \tilde{K}$.

(6) $|\theta(ghg^{-1})| < \pi$, *for all* $h \in AN$.

*Proof.* Recall (see Section 4.3.2) that the action of $AN$ on $\widetilde{\mathbb{A}^2 - 0}$ preserves all lines $(r, \ell\pi) \in \widetilde{\mathbb{A}^2 - 0}$, where $\ell \in \mathbb{Z}$, and the interior of all strips

$$\bar{\Omega}_\ell = \{(r, \theta) \mid \ell\pi \leq \theta \leq (\ell + 1)\pi\} \subset \widetilde{\mathbb{A}^2 - 0} .$$

Therefore, for any $g \in \mathrm{GL}^+(2, \mathbb{R})$ with $\theta(g) \in (\ell\pi, (\ell + 1)\pi)$ and $h \in AN$, we have

$$\ell\pi < \theta(hgh^{-1}) < (\ell + 1)\pi .$$

In particular, one deduces (5) and (6). $\qquad\qquad\qquad\qquad\qquad\qquad\square$

## A.1 The induced covering on conjugacy classes

Let $G$ be a Lie group, and

$$\mathcal{C}G = \{C(g) = \mathrm{Ad}(G)g \mid g \in G\}$$

its set of conjugacy classes. The set $\mathcal{C}G$ carries the quotient topology induced from $G$. Observe that the center of $G$ acts on $\mathcal{C}G$. Indeed, for any $z \in Z(G)$, we have $zC(g) = C(zg)$. Given a covering projection of Lie groups $\mathsf{p} : G' \to G$, there is a natural induced surjective map on conjugacy classes

$$\mathcal{C}G' \longrightarrow \mathcal{C}G , \quad C(g) \mapsto C(\mathsf{p}(g)) .$$

The kernel $\kappa$ of the covering is a central subgroup of $G'$ which acts on $\mathcal{C}G'$. As a matter of fact, $\mathcal{C}G = \mathcal{C}G'/\kappa$ is the quotient space of this action.

**Proposition A.2.**  *The natural projection map on conjugacy classes*

$$\mathcal{C}\widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \longrightarrow \mathcal{C}\mathrm{GL}^+(2,\mathbb{R})$$

*is a covering map.*

*Proof.* We consider the action of the kernel $\kappa = \langle \tau^2 \rangle$ of the covering map $\mathsf{p}$ on $\mathcal{C}\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$. For this, let $g \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ and consider its neighborhood

$$U_\epsilon = U_\epsilon(g) = \{h \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \mid |\theta(g) - \theta(h)| < \epsilon\}\ .$$

We also put

$$\mathcal{C}U_\epsilon = \{C(h) \mid h \in U_\epsilon\}$$

for the corresponding neighborhood of $C(g)$ in the space of conjugacy classes.

Let us assume first that $g \in \tilde{K}D$, $g \notin \mathcal{Z}D$. Let $V_\epsilon \subset U_\epsilon(g)$ be a neighborhood of $g$, such that all its elements are conjugate to an element of $\tilde{K} \cdot D$. Let $h \in V_\epsilon$. By using (5) of Lemma A.1, we deduce that, for all $\ell \in C(h)$,

$$|\theta(g) - \theta(\ell)| < \pi + \epsilon\ . \tag{1.1}$$

We observe that the open subsets $\mathcal{C}V_\epsilon$ and $\tau^k\mathcal{C}V_\epsilon = \mathcal{C}\tau^k V_\epsilon$ of $\mathcal{C}\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ intersect if and only if there exist elements $h, \ell \in V_\epsilon$ such that

$$\tau^k\ell \in C(h)\ .$$

If this is the case then, by the above estimate (1.1), we have

$$|\theta(g) - \theta(\tau^k\ell)| = |\theta(g) - k\pi - \theta(\ell)| < \pi + \epsilon\ . \tag{1.2}$$

Furthermore, $|\theta(g) - \theta(\ell)| < \epsilon$, since $\ell \in U_\epsilon$. If $\epsilon$ is small (1.2) is possible if and only if $k \in \{0, 1, -1\}$. For $\epsilon$ small enough, this implies that all neighborhoods of the form $\tau^{2k}\mathcal{C}V_\epsilon = \mathcal{C}\tau^{2k}V_\epsilon$ are mutually disjoint. Therefore, $\mathcal{C}V_\epsilon$ is a fundamental neighborhood of $\mathcal{C}(g)$ for the action of $\kappa$ on $\mathcal{C}\mathrm{GL}^+(2,\mathbb{R})$.

Assume next that $g \in AN$ is upper triangular. Since $g$ has real and positive eigenvalues, we may choose a small neighborhood $V_\epsilon$ as above such that all its elements are conjugate to an element of $AN$ or of $\tilde{K}D$. In particular, for all $h \in V_\epsilon$ which are conjugate to an element of $AN$, we deduce from (6) of Lemma A.1 that the range of $\theta$ on the conjugacy class $\mathcal{C}(h)$ is contained in the open interval $(-\pi, \pi)$. Consequently, $\theta(\mathcal{C}(\tau^k h))$ is contained in $((k-1)\pi, (k+1)\pi)$. It follows that all neighborhoods of the form $\tau^{2k}\mathcal{C}V_\epsilon$ are mutually disjoint, and thus $\mathcal{C}V_\epsilon$ is a fundamental neighborhood of $\mathcal{C}(g)$ for the action of $\kappa$.

An analogous argument works for $g$ with negative eigenvalues, that is, $g \in \tau AN$. Therefore $\kappa$ acts discontinuously and freely on $\mathcal{C}\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$. This implies the proposition. $\square$

**Corollary A.3.** *The natural map on conjugacy classes*

$$\mathcal{C}\widetilde{\mathrm{GL}}(2,\mathbb{R}) \longrightarrow \mathcal{C}\mathrm{GL}(2,\mathbb{R})$$

*is a local homeomorphism.*

*Proof.* Indeed, local injectivity is implied by the commutative diagram

$$
\begin{array}{ccc}
\mathcal{C}\widetilde{\mathrm{GL}}^{+}(2,\mathbb{R}) & \longrightarrow & \mathcal{C}\mathrm{GL}^{+}(2,\mathbb{R}) \\
\downarrow & & \downarrow \\
\mathcal{C}\widetilde{\mathrm{GL}}(2,\mathbb{R}) & \longrightarrow & \mathcal{C}\mathrm{GL}(2,\mathbb{R})\,.
\end{array}
$$

$\square$

**Closures of sets of conjugacy classes**    For any subset $M \subset G$ we define $\mathcal{C}M$ to be the set of conjugacy classes of elements in $M$, and $\overline{\mathcal{C}M}$ its closure in $\mathcal{C}G$. We shall require the following lemma:

**Lemma A.4.** *With the above convention the following hold in the space of conjugacy classes $\mathcal{C}\widetilde{\mathrm{GL}}^{+}(2,\mathbb{R})$:*

(1) *Let $g_i \in \widetilde{\mathrm{GL}}^{+}(2,\mathbb{R})$ be a sequence of elements such that each $g_i$ is conjugate to an element of $AN$ and such that the sequence $|\theta(g_i)|$ converges to $\pi$. Then the sequence $g_i$ leaves every compact subset of $\widetilde{\mathrm{GL}}^{+}(2,\mathbb{R})$.*

(2) $\mathcal{C}(AN) = \overline{\mathcal{C}(AN)}$ *is a closed subset of $\mathcal{C}\widetilde{\mathrm{GL}}^{+}(2,\mathbb{R})$.*

(3) $\mathcal{C}N = \overline{\mathcal{C}N} = \left\{ \mathcal{C}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \mathcal{C}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \mathrm{E}_2 \right\}$ *consists of three conjugacy classes.*

(4) $\overline{\mathcal{C}\tilde{K}} = \mathcal{C}\tilde{K} \cup \bigcup_k \mathcal{C}\,\tau^k N, \ \overline{\mathcal{C}D\tilde{K}} = D\overline{\mathcal{C}\tilde{K}}.$

*Proof.* Let $g \in \widetilde{\mathrm{GL}}^{+}(2,\mathbb{R})$ such that $|\theta(g)| = \pi$, and $\bar{g} \in \mathrm{GL}^{+}(2,\mathbb{R})$ the projection of $g$. Clearly, by definition of $\theta$, $\bar{g}$ has at least one negative eigenvalue. The sequence $g_i$ can not have a subsequence convergent to $g$, since the corresponding $\bar{g}_i$ have positive eigenvalues. Thus (1) follows.

To prove (2), we consider the subset $C(AN) \subset \widetilde{\mathrm{GL}}^{+}(2,\mathbb{R})$ which is the preimage of $\mathcal{C}(AN)$. For $g \in \widetilde{\mathrm{GL}}^{+}(2,\mathbb{R})$, let $\mathrm{dis}(g)$ denote the discriminant of the characteristic polynomial of $\mathsf{p}(g) \in \mathrm{GL}^{+}(2,\mathbb{R})$. Then $g \in C(AN)$ if and only if the following hold:

i) $|\theta(g)| < \pi$,

ii) $\mathrm{dis}(g) \geq 0$,

iii) both eigenvalues of $\mathsf{p}(g)$ are positive.

In view of (1), the condition i) is closed. Therefore, $C(AN)$ is a closed subset of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, proving (2).

Regarding (4), remark first that the closure of $\mathcal{C}K$ in $\mathcal{C}\,\mathrm{GL}^+(2,\mathbb{R})$ is contained in the union of $\mathcal{C}K$ and $\mathcal{C}N$, and $\mathcal{C}-\mathrm{E}_2N$. Now here is an example of a sequence

$$k_\varphi = \left( \begin{array}{cc} \cos\varphi + \sqrt{\sin\varphi} & -\sin\varphi - 1 \\ \sin\varphi & \cos\varphi - \sqrt{\sin\varphi} \end{array} \right)$$

of matrices, where $k_\varphi$ is conjugate to the rotation $\mathrm{R}_\varphi \in \tilde{K}$, and which, for $\varphi \to 0$ is converging to

$$\left( \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right) \in N \,.$$

Therefore, $\mathcal{C}N$ is in the closure of $\mathcal{C}\tilde{K}$. Since $\mathcal{C}\tilde{K}$ is invariant by left-multiplication with $\tau$, the same is true for its closure. This shows that $\tau^k \mathcal{C}N \subset \overline{\mathcal{C}\tilde{K}}$. $\qquad\square$

Every element $g \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ with $\mathsf{p}(g) \in AN \le \mathrm{GL}^+(2,\mathbb{R})$ is of the form $\tau^k g_o$, where $g_o \in AN \le \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$. The integer $\mathrm{lev}\, g = k \in \mathbb{Z}$ is called the *level* of $g$. The notion of level is defined for the conjugacy class $\mathcal{C}(g)$ of $g$. The following states that the level separates these conjugacy classes. In particular, the subset of conjugacy classes

$$\tau^k \mathcal{C}AN \subset \mathcal{C}\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$$

is closed.

**Proposition A.5.** *Let $g_i \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ be a sequence such that each $\mathsf{p}(g_i)$ is conjugate to an element of $AN$. If the sequence of conjugacy classes $\mathcal{C}(g_i)$ converges to $\mathcal{C}(h) \in \mathcal{C}\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ then $\mathsf{p}(h)$ is conjugate to an element of $AN$, and there exists $i_0$, such that for all $i \ge i_0$, $\mathrm{lev}\, g_i = \mathrm{lev}\, h$.*

*Proof.* The discriminant of the characteristic polynomial dis $\mathsf{p}(g_i)$ and the eigenvalues of $\mathsf{p}(g_i)$ are continuous functions on the conjugacy classes. Therefore, $\mathsf{p}(h)$ is conjugate in $\mathrm{GL}^+(2,\mathbb{R})$ to an element of $AN$. By assumption, all $\mathsf{p}(g_i)$ are contained in $\mathcal{C}AN$. Therefore, we have $g_i \in \tau^{2k_i}\mathcal{C}(h_i)$ with $h_i \in \mathcal{C}AN$. By Proposition A.2, the group generated by $\tau^2$ acts properly discontinuously on $\mathcal{C}\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$. In particular, there exists a neighbourhood $\mathcal{C}U$ of $\mathcal{C}(h)$ such that $\mathcal{C}(g_i) \in \mathcal{C}U$ implies $\mathrm{lev}\, g_i = \mathrm{lev}\, h$. $\qquad\square$

Incidentally, the assertion of Proposition A.2 fails to be true when considering the situation for the covering

$$\mathsf{Pp} : \widetilde{\mathrm{GL}}(2,\mathbb{R}) \to \mathrm{PGL}(2,\mathbb{R}) = \mathrm{GL}(2,\mathbb{R})/\{\pm\mathrm{E}_2\} \,.$$

**Example A.6.** We consider the induced map

$$\mathcal{C}\widetilde{\mathrm{GL}}(2,\mathbb{R}) \longrightarrow \mathcal{C}\mathrm{PGL}(2,\mathbb{R}) \tag{1.3}$$

on conjugacy classes. It is the quotient map of $\mathcal{C}\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ with respect to the action of the central subgroup $\mathcal{Z} = \langle \tau \rangle$, generated by the element $\tau$. Then the $\widetilde{\mathrm{GL}}(2,\mathbb{R})$-conjugacy class $\mathcal{C}(g_a)$, where $g_a \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ is a lift of

$$\bar{g}_a = \left( \begin{array}{cc} 0 & a \\ -a & 0 \end{array} \right) \in \mathrm{GL}^+(2,\mathbb{R}) \, , \ a \neq 0,$$

is fixed by translation with $\tau$. Indeed, $\tau \mathcal{C}(g_a) = \mathcal{C}(g_{-a}) = \mathcal{C}(g_a)$. Therefore, the map (1.3) cannot be a covering. It is *not even a locally injective map*: Indeed, in every neighborhood of $g_a$ there exist elements $g_{a,\epsilon}$ projecting to matrices of the form

$$\bar{g}_{a,\epsilon} = \left( \begin{array}{cc} \epsilon & a \\ -a & \epsilon \end{array} \right) \in \mathrm{GL}^+(2,\mathbb{R}) \, , \ \epsilon \neq 0.$$

Then for the conjugacy classes in $\mathrm{GL}(2,\mathbb{R})$, we have $\mathcal{C}\,\mathsf{p}(g_{a,\epsilon}) = \mathcal{C}\,\mathsf{p}(g_{-a,\epsilon})$ and therefore $\mathcal{C}\,\mathrm{P}\mathsf{p}(g_{a,\epsilon}) = \mathcal{C}\,\mathrm{P}\mathsf{p}(\tau g_{-a,\epsilon}) = \mathcal{C}\,\mathrm{P}\mathsf{p}(g_{a,-\epsilon})$. But clearly, $g_{a,\epsilon}$ and $g_{a,-\epsilon}$ are not conjugate in $\widetilde{\mathrm{GL}}(2,\mathbb{R})$ unless $\epsilon = 0$. Therefore, *the map* (1.3) *is a twofold branched covering near* $g_a$.

## A.2  Conjugacy classes of homomorphisms

Let $G$ be a Lie group. Recall that the evaluation map on the generators

$$\mathrm{Hom}(\mathbb{Z}^2, G) \to G \times G \, , \ \rho \mapsto (\rho(e_1), \rho(e_2))$$

identifies the space $\mathrm{Hom}(\mathbb{Z}^2, G)$ of all homomorphisms $\mathbb{Z}^2 \to G$ homeomorphically with an analytic subvariety of $G \times G$. With respect to this map, the orbits of the conjugation action of $G$ on $\mathrm{Hom}(\mathbb{Z}^2, G)$ correspond to sets of the form

$$C(g_1, g_2) = \{(g g_1 g^{-1}, g g_2 g^{-1}) \mid g \in G\} \subset G \times G \, .$$

We put

$$\mathcal{X}(\mathbb{Z}^2, G) = \mathrm{Hom}(\mathbb{Z}^2, G)/G$$

for the space of conjugacy classes of homomorphisms $\mathbb{Z}^2 \to G$ (also called the character variety). Given a covering homomorphism $\mathsf{p} : G' \to G$ there is a natural induced surjective map

$$\mathcal{X}(\mathbb{Z}^2, G') \longrightarrow \mathcal{X}(\mathbb{Z}^2, G) \, . \tag{1.4}$$

Returning to our specific context we introduce the following extension of Proposition A.2:

**Proposition A.7.** *The induced map on conjugacy classes of homomorphisms*

$$\mathrm{Hom}(\mathbb{Z}^2, \widetilde{\mathrm{GL}}^+(2,\mathbb{R}))/\widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \longrightarrow \mathrm{Hom}(\mathbb{Z}^2, \mathrm{GL}^+(2,\mathbb{R}))/\mathrm{GL}^+(2,\mathbb{R})$$

*is a covering map.*

*Proof.* Let $\mathrm{Z}(G)$ denote the center of $G$. This representation of $\mathrm{Hom}(\mathbb{Z}^2, G)$ as a subset of $G \times G$ gives rise to an action of $\mathrm{Z}(G) \times \mathrm{Z}(G)$ on $\mathrm{Hom}(\mathbb{Z}^2, G)$ which is determined by

$$((z_1, z_2) \cdot \rho)\,(e_i) \;=\; z_i \rho(e_i)\,,$$

where $z_i \in \mathrm{Z}(G)$. Moreover, it factors to an action of $\mathrm{Z}(G) \times \mathrm{Z}(G)$ on the space of conjugacy classes $\mathcal{X}(\mathbb{Z}^2, G)$.

Let $\kappa \leq Z(G')$ denote the kernel of $\mathsf{p} : G' \to G$. By the above, $\kappa \times \kappa$ acts on $\mathrm{Hom}(\mathbb{Z}^2, G')$, and the action factors to an action on the space of conjugacy classes $\mathcal{X}(\mathbb{Z}^2, G')$, and, as is easily verified, the natural map

$$\mathcal{X}(\mathbb{Z}^2, G')/\kappa \times \kappa \longrightarrow \mathcal{X}(\mathbb{Z}^2, G)$$

which is induced on the quotient is a homeomorphism. Therefore, 1.4 is a covering if and only if $\kappa \times \kappa$ acts discontinuously and freely on $\mathcal{X}(\mathbb{Z}^2, G')$.

Here we consider only the case $G' = \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ and $G = \mathrm{GL}^+(2,\mathbb{R})$. For any $(g_1, g_2) \in G' \times G'$, choose open neighborhoods $U_\epsilon(g_i)$ as in the proof of Proposition A.2. As follows from this previous proof, the open neighborhood

$$U_\epsilon(g_1, g_2) = U_\epsilon(g_1) \times U_\epsilon(g_2)$$

projects to a fundamental neighborhood $\mathcal{C}U_\epsilon$ for the action of $\kappa \times \kappa$ on the set of all $G'$-orbits. Hence, $\kappa \times \kappa$ acts discontinuously on $G'$-orbits. $\qquad\square$

**Corollary A.8.** *The natural map on conjugacy classes of homomorphisms*

$$\mathrm{Hom}(\mathbb{Z}^2, \widetilde{\mathrm{GL}}^+(2,\mathbb{R}))/\widetilde{\mathrm{GL}}(2,\mathbb{R}) \longrightarrow \mathrm{Hom}(\mathbb{Z}^2, \mathrm{GL}^+(2,\mathbb{R}))/\mathrm{GL}(2,\mathbb{R})$$

*is a local homeomorphism.*

# B  Example of a two-dimensional geometry where *hol* is not a local homeomorphism

Let $(X, G)$ be the homogeneous geometry which is defined by the natural action of $\mathrm{PGL}(2,\mathbb{R}) = \mathrm{GL}(2,\mathbb{R})/\{\pm 1\}$ on the space

$$X = \mathrm{P}(\mathbb{A}^2 - 0) = \mathbb{A}^2 - 0/\{\pm 1\}\,,$$

that is, $X$ is the quotient space of $\mathbb{R}^2 - \{0\}$ by the action of the center $\{E_2, -E_2\}$ of $\mathrm{SL}(2, \mathbb{R})$. The natural map

$$(\mathbb{A}^2 - 0, \mathrm{GL}(2, \mathbb{R})) \longrightarrow (\mathrm{P}(\mathbb{A}^2 - 0), \mathrm{PGL}(2, \mathbb{R}))$$

is a covering of geometries in the sense of Definition 3.7. By Lemma 3.22, the induced map on deformation spaces

$$\mathfrak{D}(T^2, \mathbb{A}^2 - 0) \longrightarrow \mathfrak{D}(T^2, \mathrm{P}(\mathbb{A}^2 - 0)) \tag{2.1}$$

is a homeomorphism.

We claim that the *holonomy for the deformation space* $\mathfrak{D}(T^2, \mathrm{P}(\mathbb{A}^2 - 0))$ *is* not *a local homeomorphism.* For this we recall first the $(\mathbb{A}^2 - 0, \mathrm{GL}(2, \mathbb{R}))$-manifolds $\mathcal{H}_{\lambda, \frac{\pi}{2}, k}$ constructed in Example 4.10 (finite quotients of Hopf tori). Then we observe:

**Proposition B.1.** *The holonomy map*

$$\mathfrak{D}(T^2, \mathrm{P}(\mathbb{A}^2 - 0)) \xrightarrow{\ hol\ } \mathrm{Hom}(\mathbb{Z}^2, \mathrm{PGL}(2, \mathbb{R}))/\mathrm{PGL}(2, \mathbb{R})$$

*is a twofold branched covering near the image of a homogeneous flat affine torus* $\mathcal{H}_{\lambda, \frac{\pi}{2}, k}$ *under the map* (2.1).

*Proof.* The commutative diagram (3.8) for the subgeometry takes the form

$$
\begin{array}{ccc}
\mathfrak{D}(T^2, \mathbb{A}^2 - 0) & \xrightarrow{\ hol\ } & \mathrm{Hom}(\mathbb{Z}^2, \mathrm{GL}^+(2, \mathbb{R}))/\,\mathrm{GL}(2, \mathbb{R}) \\
\approx \downarrow & & \downarrow \\
\mathfrak{D}(T^2, \mathrm{P}(\mathbb{A}^2 - 0)) & \xrightarrow{\ hol\ } & \mathrm{Hom}(\mathbb{Z}^2, \mathrm{PGL}(2, \mathbb{R}))/\mathrm{PGL}(2, \mathbb{R})\,.
\end{array}
$$

Note that, by Corollary 6.10, the top horizontal map is a local homeomorphism. Furthermore, by Example A.6, the right vertical map is a twofold branched covering near the holonomy homomorphism of every flat affine torus $\mathcal{H}_{\lambda, \frac{\pi}{2}, k}$. We deduce that the bottom map *hol* for $\mathfrak{D}(T^2, \mathrm{P}(\mathbb{A}^2 - 0))$ is locally a twofold branched covering at the images of $\mathcal{H}_{\lambda, \frac{\pi}{2}, k}$. $\qquad\square$

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